Math 2270-6

Notes of 2/26/2019

Isomorphisms

The main point today is that in a profound sense all finite (n-)dimensional vector spaces are essentially indistinguishable from $\mathbb{R}^n$.

- Start with an example. Let

$$V = \{p : p(x) = ax^2 + bx + c\}$$

the linear space of all quadratic polynomials.

- Moreover consider the map (function) C from V to $\mathbb{R}^3$ defined by

$$ax^2 + bx + c \mapsto \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

- For example, if

$$p(x) = x^2 + 2x + 3$$

then

$$C(p) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

- We can do a lot with polynomials. We can differentiate, integrate, substitute, multiply, change variables, compose … them.

However, as far as addition and multiplication with scalars works there is no difference—other than notation—between $\mathbb{R}^3$ and V.

- As far as addition and scalar multiplication is concerned there is nothing in V that isn't all captured in $\mathbb{R}^3$.
- We say that V and $\mathbb{R}^3$ are **isomorphic**.

- The map

$$ax^2 + bx + c \mapsto \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

  has three key properties:

1. It is **onto**.
2. It is **one-to-one**.
3. It is **linear**.

- Moreover, as far as the vector properties of quadratic polynomials are concerned, it does not matter whether we operate on a polynomial or on its image in $\mathbb{R}^3$.
- As far as adding polynomials and multiplying them with scalars (as opposed to, for example, integrating, differentiating, or composing them) it really does not matter whether we do these operations in V or in $\mathbb{R}^3$.
- technically, we say that V and $\mathbb{R}^3$ are **isomorphic.**
- **Definition:** Two vector spaces V and W are **isomorphic** if there exists a linear function

  $$C : V \longrightarrow W$$

  that is one-to-one and onto.
- In that case, C is called an **isomorphism**.

- Suppose V and W are vector spaces of the same dimension, say n, the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis of V, and the set $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$ is a basis of W. Then the function

  $$C : \sum_{i=1}^{n} \alpha_i \mathbf{v}_i \quad \mapsto \quad \sum_{i=1}^{n} \alpha_i \mathbf{w}_i$$

  is an isomorphism.

- Isomorphisms are invertible, and their inverses are isomorphisms as well.

- The previous result shows that any two vector spaces of the same dimension are isomorphic. It works the other way too. If two vector spaces are not isomorphic then they must have different dimensions.

- To see this suppose that V has dimension n, M has dimension m, and $m > n$. Also suppose $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis of V, and C is an isomorphic from V to W. Then since $\mathcal{B}$ is a spanning set of V and C is onto, the set
$$C(\mathbf{v}_1), C(\mathbf{v}_2), \ldots, C(\mathbf{v}_n)$$
must be a spanning set of W. But it can't be because it does not contain enough vectors.

- As an exercise, you may want to think about $\mathbb{R}^n$ and $\mathbb{R}^m$ and show they are not isomorphic expressing a proposed isomorphism as a matrix.

Coordinates

- Suppose V is an n-dimensional vector space and the set

$$\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$$

is a basis of V. Then every vector $\mathbf{v} \in V$ can be expressed uniquely as a linear combination

$$\mathbf{x} = \sum_{i=1}^{n} \alpha_i \mathbf{v}_i$$

of the basis vectors. Then the vector

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$$

is called the **coordinate vector of x with respect to** $\mathcal{B}$ and the coefficients α_i are the **coordinates of x** with respect to $\mathcal{B}$.

- The **coordinate mapping**

$$\mathbf{x} \quad \mapsto \quad [\mathbf{x}]_{\mathcal{B}}$$

is an isomorphism.

- Since all n-dimensional vector spaces are isomorphic to $\mathbb{R}^n$ we can express linear operations between them as matrices.
- Example: Consider the space of all quadratic polynomials. Express differentiation and integrations as a matrices.

- We have learned how to express coordinates and linear operations with respect to any basis. There are often natural choices, but others are possible.