For the last two lectures we’ve studied first-order differential equations in standard form

\[ y' = f(x, y). \]

We learned how to solve these differential equations for the special situation where \( f(x, y) \) is independent of the variable \( y \), and is just a function of \( x \), \( f(x) \). We also learned about slope fields, which give us a geometric method for understanding solutions and approximating them, even if we cannot find them directly.

Today we’re going to discuss how to solve first-order differential equations in standard form in the special situation where the function \( f(x, y) \) is separable, which means we can write \( f(x, y) \) as the product of a function of \( x \), and a function of \( y \).

The exercises for this section are:

Section 1.4 - 1, 3, 17, 19, 31, 35, 53, 68
Separable Equations and How to Solve Them

Suppose we have a first-order differential equation in standard form:

\[ \frac{dy}{dx} = h(x, y). \]

If the function \( h(x, y) \) is separable we can write it as the product of two functions, one a function of \( x \), and the other a function of \( y \). So,

\[ h(x, y) = \frac{g(x)}{f(y)}. \]

In this situation we can manipulate our differential equation to put everything with a \( y \) term on one side, and everything with an \( x \) term on the other:

\[ f(y)\,dy = f(x)\,dx. \]

From here we can just integrate both sides of the equation, and then solve for \( y \) as a function of \( x \)!

So, for example, suppose we’re given the differential equation

\[ \frac{dP}{dt} = p^2. \]

We can rewrite this equation as

\[ \frac{dP}{p^2} = dt, \]

and then integrate both sides of the equation to get

\[ -\frac{1}{p} = t + C. \]
Solving this for \( P \) as a function of \( t \) gives us

\[
P(t) = \frac{1}{C - t}. \quad ^1
\]

Note that this function has a vertical asymptote as \( t \) approaches \( C \). If this is a population model, this is called doomsday!

**Examples of Separable Differential Equations**

Suppose we’re given the differential equation

\[
\frac{dy}{dx} = \frac{4 - 2x}{3y^2 - 5}.
\]

This differential equation is separable, and we can rewrite it as

\[
(3y^2 - 5)dy = (4 - 2x)dx.
\]

If we integrate both sides of this differential equation

\[
\int (3y^2 - 5)dy = \int (4 - 2x)dx
\]

we get

\[
y^3 - 5y = 4x - x^2 + C.
\]

This is a solution to our differential equation, but we cannot readily solve this equation for \( y \) in terms of \( x \). So, our solution to this differential equation must be implicit.

\(^1\)Note that we’re playing a little fast and loose with the unknown constant \( C \) here. In particular, if we multiply an unknown constant \( C \) by \(-1\), it’s still just an unknown constant, and we continue to call it (positive) \( C \).
If we’re given an initial value, say \( y(1) = 3 \), then we can easily solve for the unknown constant \( C \):

\[
3^3 - 5(3) = 4(1) - 1^3 + C \Rightarrow C = 9.
\]

So, around the point \((1, 3)\) the differential equation will have the unique solution given implicitly by the curve defined by

\[
y^3 - 5y = 4x - x^5 + 9.
\]

**Example** - Find all solutions to the differential equation

\[
\frac{dy}{dx} = 6x(y - 1)^{\frac{2}{3}}.
\]

\[
\int \frac{dy}{(y-1)^{\frac{1}{3}}} = \int 6x \, dx
\]

\[
3 \, (y-1)^{\frac{1}{3}} = 3x^2 + C
\]

\[
\Rightarrow (y-1)^{\frac{1}{3}} = x^2 + C
\]

\[
\Rightarrow y = (x^2 + C)^3 + 1
\]

Now, the function \( y(x) = 1 \) is also a solution!

If we’re given the initial value problem \( y(0) = 1 \) then we have 2 solutions:

\( y_1(x) = x^6 + 1 \) and \( y_2(x) = 1 \).

So, what’s going on here? Turn the page to find out.
More room for the example.

The function

\[ f(x,y) = 6x (y-1)^{3/5} \]

is continuous everywhere, but

\[ \frac{\partial f}{\partial y} = \frac{4x}{(y-1)^{1/5}} \]

is undefined where \( y = 1 \). So, for any initial value \( y(a) = b \) if \( b \neq 1 \) there is a unique local solution, but if \( b = 1 \) there is not.
A very common, and simple, type of differential equation that is used to model many, many things\(^2\) is

\[
\frac{dx}{dt} = kx
\]

where \(k\) is some constant.

Now, this is a separable equation, and so it can be solved by our methods. First, we rewrite it as

\[
\frac{dx}{x} = kdt,
\]

and then integrate both sides

\[
\int \frac{dx}{x} = \int kdt
\]

to get

\[
\ln x = kt + C.
\]

If we then exponentiate both sides we get

\[
x(t) = e^{kt+C} = e^C e^{kt} = Ce^{kt}.\]

So, the solution to our differential equation is exponential growth (if \(k > 0\)) or exponential decay (if \(k < 0\)). If \(k = 0\) the answer is just a boring unknown constant.

\(^2\)Compound interest, population growth, radioactive decay, etc...

\(^3\)The American Society for the Prevention of Notation Abuse would strongly protest this last equality. I'm just saying that \(e^C\), where \(C\) is an unknown constant, is itself just an unknown constant, and I don't like having to come up with new letters, so I just continue to represent the unknown constant as \(C\).
Radioactive decay is quite accurately measured by an exponential decay function. For $^{14}C$ decay, the decay constant is $k \approx -0.0001216$ if $t$ is measured in years.

**Example** - Carbon taken from a purported relic of the time of Christ contained $4.6 \times 10^{10}$ atoms of $^{14}C$ per gram. Carbon extracted from a present-day specimen of the same substance contained $5.0 \times 10^{10}$ atoms of $^{14}C$ per gram. Compute the approximate age of the relic. What is your opinion as to its authenticity?

\[
C(t) = C_0 e^{kt}
\]

\[
C(t) = 4.6 \times 10^{10}
\]

\[
C_0 = 5.0 \times 10^{10}
\]

\[
k = -0.0001216
\]

\[
\Rightarrow t_0 = \ln \left( \frac{4.6 \times 10^{10}}{5.0 \times 10^{10}} \right) \approx 685.7 \text{ years.}
\]

So, probably not from the time of Christ (about 2,000 years ago.)