

Math 2270 - Lecture 33 : Positive Definite Matrices

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This lecture covers *section 6.5* of the textbook.

Today we're going to talk about a special type of symmetric matrix, called a *positive definite* matrix. A positive definite matrix is a symmetric matrix with all positive eigenvalues. Note that as it's a symmetric matrix all the eigenvalues are real, so it makes sense to talk about them being positive or negative.

Now, it's not always easy to tell if a matrix is positive definite. Quick, is this matrix?

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

Hard to tell just by looking at it.¹

One way to tell if a matrix is positive definite is to calculate all the eigenvalues and just check to see if they're all positive. The only problem with this is, if you've learned nothing else in this class, you've probably learned that calculating eigenvalues can be a real pain. Especially for large matrices. So, today, we're going to learn some easier ways to tell if a matrix is positive definite.

The assigned problems for this section are:

Section 6.5 - 2, 3, 7, 11, 16

¹It's not.

1 Positive Definite Matrices - What Are They, and What Do They Want?

I've already told you what a positive definite matrix is.

A matrix is positive definite if it's symmetric and all its eigenvalues are positive.

The thing is, there are a lot of other equivalent ways to define a positive definite matrix. One equivalent definition can be derived using the fact that for a symmetric matrix the signs of the pivots are the signs of the eigenvalues. So, for example, if a 4×4 matrix has three positive pivots and one negative pivot, it will have three positive eigenvalues and one negative eigenvalue. This is proven in section 6.4 of the textbook. We can apply this fact to positive definite matrices to derive the next equivalent definition.

A matrix is positive definite if it's symmetric and all its pivots are positive.

Pivots are, in general, *way* easier to calculate than eigenvalues. Just perform elimination and examine the diagonal terms. No problem. In practice this is usually the way you'd like to do it. For example, in that matrix from the introduction

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

If we perform elimination (subtract $2 \times$ row 1 from row 2) we get

$$\begin{pmatrix} 1 & 2 \\ 0 & -3 \end{pmatrix}$$

The pivots are 1 and -3 . In particular, one of the pivots is -3 , and so the matrix is not positive definite. Were we to calculate the eigenvalues we'd see they are 3 and -1 .

Another way we can test for if a matrix is positive definite is we can look at its n upper left determinants. Remember from section 5.2 that the k th pivot of a matrix is

$$d_k = \frac{\det(A_k)}{\det(A_{k-1})}$$

where A_k is the upper left $k \times k$ submatrix. All the pivots will be positive if and only if $\det(A_k) > 0$ for all $1 \leq k \leq n$. So, if all upper left $k \times k$ determinants of a symmetric matrix are positive, the matrix is positive definite.

Example - Is the following matrix positive definite?

$$2 \quad \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

$$\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3$$

$$\begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix} = 8 - 2 - 2 = 4$$

$2, 3, 4 > 0 \Rightarrow$ Positive definite.

If \mathbf{x} is an eigenvector of A then $\mathbf{x} \neq 0$ and $A\mathbf{x} = \lambda\mathbf{x}$. In this case $\mathbf{x}^T A\mathbf{x} = \lambda\mathbf{x}^T \mathbf{x}$. If $\lambda > 0$, then as $\mathbf{x}^T \mathbf{x} > 0$ we must have $\mathbf{x}^T A\mathbf{x} > 0$.

The thing about positive definite matrices is $\mathbf{x}^T A \mathbf{x}$ is *always* positive, for any non-zero vector \mathbf{x} , not just for an eigenvector.² In fact, this is an equivalent definition of a matrix being positive definite.

A matrix is positive definite if $\mathbf{x}^T A \mathbf{x} > 0$ for all vectors $\mathbf{x} \neq 0$.

Frequently in physics the *energy* of a system in state \mathbf{x} is represented as $\mathbf{x}^T A \mathbf{x}$ (or $\frac{1}{2} \mathbf{x}^T A \mathbf{x}$) and so this is frequently called the *energy-based* definition of a positive definite matrix.

This definition makes some properties of positive definite matrices much easier to prove.

Example - Prove if A and B are positive definite then so is $A + B$.

$$\vec{x}^T (A+B) \vec{x} = \vec{x}^T A \vec{x} + \vec{x}^T B \vec{x}$$

$$\Rightarrow > 0$$

for all $\vec{x} \neq 0$,

So, $A+B$ is positive definite.

Our final definition of positive definite is that a matrix A is positive definite if and only if it can be written as $A = R^T R$, where R is a matrix, possibly rectangular, with independent columns. Note that, using our energy-based definition, it's easy to prove that if $A = R^T R$ then A is positive definite.

²The reason for this is that the eigenvectors of a symmetric $n \times n$ matrix span all of \mathbb{R}^n , and so any vector can be represented as a linear combination of the eigenvectors. We haven't proven this yet, but that's why.

$$\mathbf{x}^T \mathbf{A} \mathbf{x} = \mathbf{x}^T \mathbf{R}^T \mathbf{R} \mathbf{x} = (\mathbf{R} \mathbf{x})^T (\mathbf{R} \mathbf{x}) = \|\mathbf{R} \mathbf{x}\|^2.$$

If the columns of \mathbf{R} are linearly independent then $\mathbf{R} \mathbf{x} \neq \mathbf{0}$ if $\mathbf{x} \neq \mathbf{0}$, and so $\mathbf{x}^T \mathbf{A} \mathbf{x} > 0$.

A matrix \mathbf{A} is positive definite if and only if it can be written as $\mathbf{A} = \mathbf{R}^T \mathbf{R}$ for some possibly rectangular matrix \mathbf{R} with independent columns.

Note that we say a matrix is *positive semidefinite* if all of its eigenvalues are non-negative.

Example - For what numbers b is the following matrix positive semidefinite?

$$\begin{pmatrix} 2 & -1 & b \\ -1 & 2 & -1 \\ b & -1 & 2 \end{pmatrix}$$

$$\begin{vmatrix} 2 & -1 & b \\ -1 & 2 & -1 \\ b & -1 & 2 \end{vmatrix} = 8 + b + b - 2 - 2 - 2b^2 \\ = -2b^2 + 2b + 4 \geq 0$$

$$\Rightarrow b^2 - b - 2 \leq 0$$

$$(b-2)(b+1) \leq 0$$

$$\boxed{-1 \leq b \leq 2}$$