Math 2270 - Lecture 11: Transposes and Permutations

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This lecture covers **section 2.7** of the textbook.

1 Transposes

The transpose of a matrix is the matrix you get when you switch the rows and the columns. For example, the transpose of

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{pmatrix}$$

is the matrix

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 4 \end{pmatrix}$$

We denote the transpose of a matrix A by A^T. Formally, we define

$$(A^T)_{ij} = A_{ji}$$

Example - Calculate the transposes of the following matrices

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \qquad \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} \qquad \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}^T = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix} \qquad \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}^T = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}^T = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$$

The transpose of the sum of two matrices is the sum of the transposes

$$(A+B)^T = A^T + B^T$$

which is pretty straightforward. What is less straightforward is the rule for products

$$(AB)^T = B^T A^T$$

The book has a proof of the above. Check it out. Another proof is to just look at the definition of matrix products and note

$$(AB)^T_{ij} = AB_{ji} = \sum_k A_{jk} B_{ki} = \sum_k B_{ki} A_{jk} = \sum_k B^T_{ik} A^T_{kj} = (B^T A^T)_{ij}$$

The transpose of the identity matrix is still the identity matrix $I^T = I$. Knowing this and using our above result it's quick to get the transpose of an inverse

$$AA^{-1} = I = I^T = (AA^{-1})^T = (A^{-1})^T A^T$$

So, the inverse of A^T is $(A^{-1})^T$. Stated otherwise $(A^T)^{-1} = (A^{-1})^T$. In words, the inverse of the transpose is the transpose of the inverse.

Example - Find A^T and A^{-1} and $(A^{-1})^T$ and $(A^T)^{-1}$ for

$$A = \begin{pmatrix} 1 & 0 \\ 9 & 3 \end{pmatrix}$$

$$A^T = \begin{pmatrix} 1 & 9 \\ 0 & 3 \end{pmatrix} \qquad (A^T)^{-1} = \begin{pmatrix} 1 & -3 \\ 0 & \frac{1}{3} \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} 1 & 0 \\ -3 & \frac{1}{3} \end{pmatrix} \qquad (A^{-1})^T = \begin{pmatrix} 1 & -3 \\ 0 & \frac{1}{3} \end{pmatrix}$$

2 Symmetric Matrices

A symmetric matrix is a matrix that is its own transpose. Stated slightly more mathematically, a matrix A is symmetric if $A = A^T$. Note that, obviously, all symmetric matrices are square matrices.

For example, the matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 4 & 1 \end{pmatrix}$$

is symmetric. Note $(A^{-1})^T = (A^T)^{-1} = A^{-1}$, so the inverse of a symmetric matrix is itself symmetric.

For any matrix, square or not, we can construct a *symmetric product*. There are two ways to do this. We can take the product $R^T R$, or the product RR^T. The matrices $R^T R$ and RR^T will both be square and both be symmetric, but will rarely be equal. In fact, if R is not square, the two will not even be the same size.

We can see this in the matrix

$$R = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

The two symmetric products are

$$RR^T = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

$$R^T R = \begin{pmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

These two symmtric products are unequal[1], but both are symmetric. Also, note that none of the diagonal terms is negative. This is not a coincidence.

[1]They're not even the same size!

Example - Why are all diagonal terms on a symmetric product non-negative?

A diagonal term is of the form

$(RR^T)_{ii}$ = (row i of R)$\cdot$(column i of R^T)

As (row i of R) = (column i of R^T)

this is the dot product of a vector with itself, which is always non-negative

Returning to the theme of the last lecture, if A is symmetric then the LDU factorization $A = LDU$ has a particularly simple form. Namely, if $A = A^T$ then $U = L^T$ and $A = LDL^T$.

Example - Factor the following matrix into $A = LDU$ form and verify $U = L^T$

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 7 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}^T = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

3 Permutation Matrices

A permutation matrix is a square matrix that rearranges the rows of another matrix by multiplication. A permutation matrix P has the rows of the identity I in any order. For $n \times n$ matrices there are $n!$ permutation matrices. For example, the matrix

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Puts row 3 in row 1, row 1 in row 2, and row 2 in row 3. In cycle notation[2] we'd represent this permutation as (123).

Example - What is the 3×3 permutation matrix that switches rows 1 and 3?

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Now, if you recall from elimination theory we sometime have to switch rows to get around a zero pivot. This can mess up our nice $A = LDU$ form. So, we usually assume we've done all the permutations we need to do before we start elimination, and write this as $PA = LDU$, where P is a permutation matrix such that elimination works. The book mentions this, but says not to worry too much about it. I agree.

[2]Don't worry if you don't know what that means.