Chapter 8

Markowitz Portfolio Theory

8.7 Investor Utility Functions

People are always asked the question: would more money make you happier? The answer is usually yes. The next question is how much more money you would need to let you feel noticeably happier? a lot or just a little. The utility function for an investor is introduced to describe the behavior related to these two questions. For the obvious reasons, it is also called the happiness function. More specifically, a utility function $u(x)$ is a function of wealth (x) that quantifies the happiness level. The first observation made above is translated in mathematical terms as

$$u'(x) > 0$$

and if you believe in diminishing return of the wealth, then you would require

$$u''(x) < 0.$$

In real life, the future wealth level will be uncertain, so we would use a random variable X as the input, and we are often involved in the study of the expectation. First of all, we notice that unless u is linear,

$$\mathbb{E}[u(X)] \neq u\left(\mathbb{E}[X]\right)$$

So there is a question that if one should maximize the one on the left, or the one on the right? The answer to this question for most investors is quite clear: it is the utility (happiness for ordinary people) that investors should care for. Therefore, the utility theory suggest that we should prefer a random variable such that $\mathbb{E}[u(X)]$ is maximized!

Suppose that $u'' < 0$, Jensen's inequality gives

$$\mathbb{E}[u(X)] < u\left(\mathbb{E}[X]\right)$$

We can actually show this (not rigorously) using a simple Taylor expansion around the value $\mathbb{E}[X]$:

$$u(X) = u\left(\mathbb{E}[X]\right) + u'\left(\mathbb{E}[X]\right)\left(X - \mathbb{E}[X]\right) + \frac{1}{2}u''\left(\mathbb{E}[X]\right)\left(X - \mathbb{E}[X]\right)^2 + \cdots$$

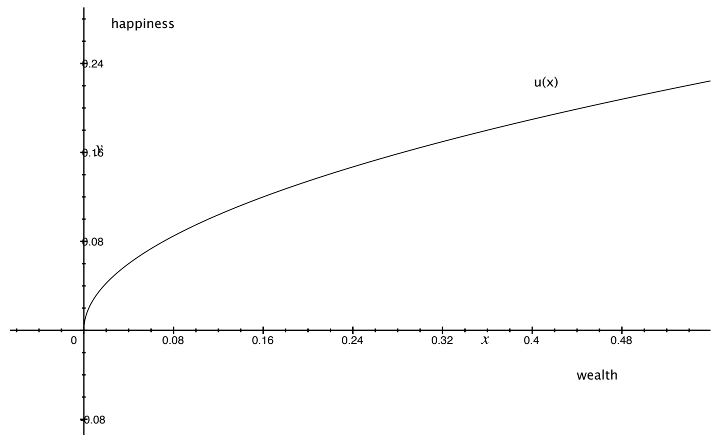


Figure 8.1: A simple example of the utility function.

If we take the expectation of both sides, assuming $u'' < 0$,

$$\begin{aligned}
\mathbb{E}[u(X)] &= u\left(\mathbb{E}[X]\right) + u'\left(\mathbb{E}[X]\right)\mathbb{E}\left[X - \mathbb{E}[X]\right] + \frac{1}{2}u''\left(\mathbb{E}[X]\right)\mathbb{E}(X - \mathbb{E}[X])^2 + \cdots \\
&= u\left(\mathbb{E}[X]\right) + \frac{1}{2}u''\left(\mathbb{E}[X]\right)\mathrm{Var(X)} + \cdots \\
&< u\left(\mathbb{E}[X]\right)
\end{aligned}$$

If we let X to be the realized return of certain investment, the investors' goal would be to maximize the expected utility, and from the last equation above it seems that we should **maximize** the expected value of the return ($\mathbb{E}[X]$ and **minimize** the variance of the return (Var(X)) at the same time. This is what is contained in Markowitz portfolio theory. But this result shows that optimizing the return of the portfolio can be viewed as maximizing the utility for the investor. The advantage of the utility approach is that now we can use different utility functions for different investors' preferences, mostly about their preferences towards risk. Here is one example of utility functions.

Example 1 *We assume $u(x) = 1 - e^{-x}$ and X is a normal random variable. Then we find*

$$\mathbb{E}[u(X)] = 1 - \mathbb{E}[e^{-X}] = 1 - e^{-\mathbb{E}[X] + \frac{1}{2}\mathrm{Var}(X)}$$

Since the exponential is an increasing function, the following are equivalent:

$$\max \mathbb{E}[u(X)]$$

$$\min e^{-\mathbb{E}[X] + \frac{1}{2}\mathrm{Var}(X)}$$

$$\min \left[-\mathbb{E}[X] + \frac{1}{2}\mathrm{Var}(X)\right]$$

$$\max \left[\mathbb{E}[X] - \frac{1}{2}\mathrm{Var}(X) \right]$$

This is achieved when the expectation of X is high and the variance of X is low, with some balance. If we modify the exponential term to e^{-ax} with $a > 0$, we can modify the balance bias by choosing the parameter a carefully.

How do we interpret the condition $u'' < 0$?
Consider the following two investment opportunities with two respective scenarios for the returns:

Security A has a riskless return of 10%, and security B has a risky return of 15% or 5%, depending on which scenario. Their expected values are the same, but

$$\mathbb{E}\left[u(R^A)\right] \neq \mathbb{E}\left[u(R^B)\right]$$

for most utility functions. With approximations, we can establish

$$\mathbb{E}\left[u(R)\right] \approx u\left(\mathbb{E}[R]\right) + \frac{1}{2}u''(\mathbb{E}[R])\mathrm{Var}(R)$$

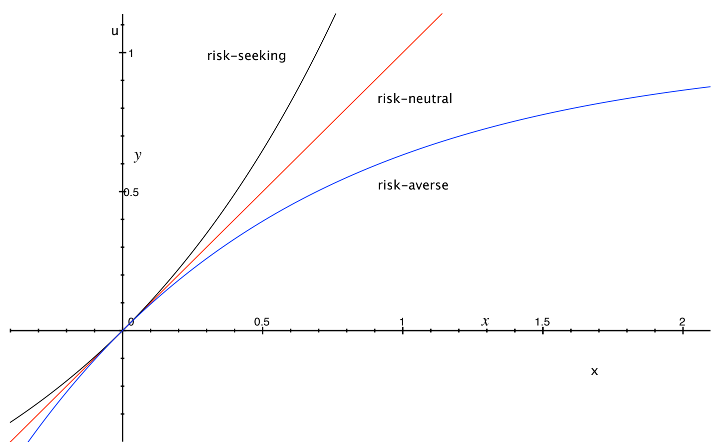


Figure 8.2: Shapes of utility functions to distinguish investors risk preferences

Using the above example, notice that $\mathbb{E}[R^A] = \mathbb{E}[R^B], \;\; \mathrm{Var}(R^A) = 0, \mathrm{Var}(R^B) < 0$, we can summarize the risk preference of the investor via utility functions as follows.

1. If $u'' < 0$, $\mathbb{E}\left[u(R^B)\right] < \mathbb{E}\left[u(R^A)\right]$, so the investor prefers A to B, so he/she is risk-averse.

2. If $u'' = 0$, $\mathbb{E}\left[u(R^B)\right] = \mathbb{E}\left[u(R^A)\right]$, so the investor is indifferent to A and B, so he/she is risk-neutral.

3. If $u'' > 0$, $\mathbb{E}\left[u(R^B)\right] > \mathbb{E}\left[u(R^A)\right]$, so the investor prefers B to A, so he/she is risk-seeking.

8.8 Capital Market Theory

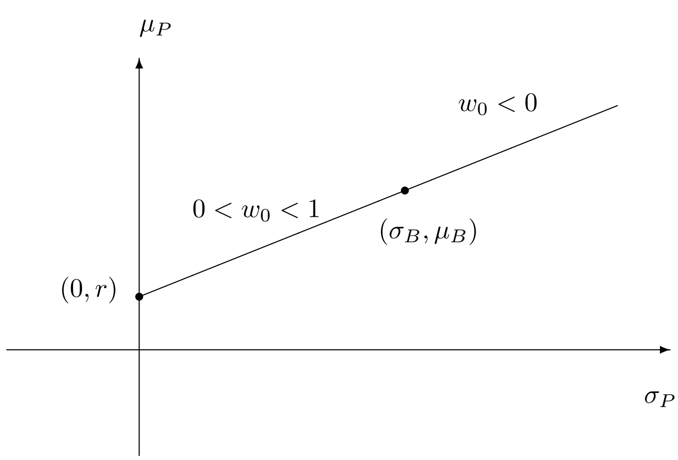
In the Markowitz portfolio theory presented, there is an assumption that all of the securities have $\sigma > 0$, which excludes the choice for a risk-free security, such as the treasury bond. Of course most investors would like to include a risk-free component in the portfolio, and an obvious solution is to combine a minimum variance portfolio represented by (σ_B, μ_B) with a risk-free asset represented by $(0, r)$:

$$N \text{ risky assets } + \text{ 1 risk-free asset}$$

and the combined portfolio will have its risk-return pair

$$(x, y) = w_0(0, r) + (1 - w_0)(\sigma_B, \mu_B)$$

By varying w_0 we end up all the possible portfolios with their risk-return on the straight line shown in the following graph.



It is natural to choose the risky portfolio to be on the frontier, and even more specifically, we should choose the portfolio that corresponds to the point on the frontier that leads to the tangent line when we connect it with $(0, r)$. This portfolio is indeed very special and we call it the *market portfolio* (σ_M, μ_M). The straight line in this case is called the *capital market line* or CML. This combination using $(0, r)$ and (σ_M, μ_M) gives us all the points on the CML and they represent an optimized combination. To see that the CML represents the optimal combination, just perturb the straight line in both directions and see what will happen. If you move the straight line up, then there is no point within the Markowitz bullet, which means the risky portfolio cannot be constructed. On the other hand, if we move the straight line down, there will be a secant line with the bullet, which gives a set of portfolios that are not on the frontier. That would suggest that this risky portfolio can be improved, therefore it is not optimal. With these arguments we conclude that the combination with the risky portfolio is the optimal combination.

Example 2 *We assume that* $r = 6\%$, $\mu_M = 12\%$. *If we choose* $w_0 = 0.75$, *we have a combination portfolio with expected return* $\mu = 0.75 \times 0.06 + 0.25 \times 0.12 = 7.5\%$. *If we choose* $w_0 = -0.75$, *we have a combination portfolio with expected return* $\mu = -0.75 \times 0.06 + 1.75 \times 0.12 = 16.5\%$. *The second case represents a leverage, meaning the investor is using borrowed money to invest in the risky market portfolio. The expected return is substantially improved but it comes with an increased risk level* σ.

The market portfolio weights can be obtained by solving for the tangent line, and the solutions are given by

$$\mu_M = \frac{C - Br}{B - Ar}, \quad \sigma_M^2 = \frac{Ar^2 - 2Br + C}{(B - Ar)^2}, \quad \mathbf{w}_M = \frac{V^{-1}(\boldsymbol{\mu}_M - r\mathbf{e})}{B - Ar}$$

Here $\boldsymbol{\mu}_M$ is the expected return vector for all those securities that are listed in the market portfolio, and $\mathbf{e} = (1, 1, \ldots, 1)^T$. An example is given in figure 8.3.
In case $r = 0$, we can see that

$$\mu_M = \mu_D, \quad \sigma_M = \sigma_D$$

That is, the market portfolio is reduced to the diversified portfolio in case $r = 0$. This explains why we called (σ_D, μ_D) the diversified portfolio, as the market portfolio is supposed to best express the market information which is well diversified.

8.9 Capital Asset Pricing Model (CAPM)

The CAPM model in finance is a major milestone and it focuses on the particular relation between the individual asset (i) and the market. We begin with the introduction of the security's *beta*, that measures the degree to which the security's return moves in accordance with the market's return:

$$\beta_i = \frac{\mathrm{Cov}(R_i, R_M)}{\sigma_M^2} = \rho_{iM} \frac{\sigma_i}{\sigma_M} \tag{8.1}$$

If we can estimate β for a particular security, we can obtain other important information about the security. For example, we can have a lower bound for σ_i:

$$\sigma_i \geq |\beta_i| \sigma_M$$

How does the β help us in pricing a security?

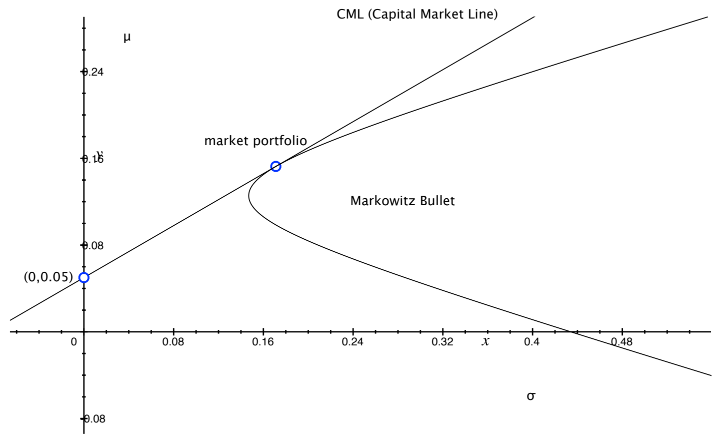


Figure 8.3: The Capital Market Line for a 3-Security Portfolio

Theorem 1 *(Capital Asset Pricing Theorem) Assume that the covariance matrix V is positive definite, $\boldsymbol{\mu}$ and $\mathbf{e}$ are linearly independent, then*

$$\mu_i - r = \beta_i(\mu_M - r) \tag{8.2}$$

Proof: We need to show

$$\beta_i = \frac{\mathbf{e}_i^T V \mathbf{w}_M}{\mathbf{w}_M^T V \mathbf{w}_M} = \frac{\mu_i - r}{\mu_M - r}$$

First we compute the covariance

$$\mathrm{Cov}(R_i, R_M) = \mathrm{Cov}(R_i, \mathbf{w}_M^T \mathbf{R}_M) = \mathrm{Cov}(R_i, \sum_{l=1}^{N} w_l^M R_l) = \sum_{l=1}^{N} w_l^M \mathrm{Cov}(R_i, R_l) = \mathbf{e}_i^T V \mathbf{w}_M$$

So

$$\beta_i = \frac{\mathbf{e}_i^T V \mathbf{w}_M}{\mathbf{w}_M^T V \mathbf{w}_M}$$

But

$$\mathbf{w}_M = \frac{V^{-1}(\boldsymbol{\mu}_M - r\mathbf{e})}{B - Ar}$$

Then

$$\mathbf{e}_i^T V \mathbf{w}_M = \mathbf{e}_i^T \frac{\boldsymbol{\mu}_M - r\mathbf{e}}{B - Ar} = \frac{\mu_i - r}{B - Ar} \tag{8.3}$$

$$\mathbf{w}_M^T V \mathbf{w}_M = \mathbf{w}_M^T \frac{\boldsymbol{\mu}_M - r\mathbf{e}}{B - Ar} = \frac{\mu_M - r}{B - Ar} \tag{8.4}$$

so finally

$$\beta_i = \frac{\mu_i - r}{\mu_M - r} \tag{8.5}$$

We have the following interpretations:

- The slope of the line show how much the particular security is correlated with the market.
- $\beta(\mu_M - r)$ is the risk premium.
- The sign of β explains whether the security and market move more likely in the same or the opposite directions.

Why is the CAPM model called a pricing model? It turns out that with the information about β, this model can be useful in pricing the current value of an asset. Consider the return of the asset over a time period

$$R = \frac{S_T - S_0}{S_0}$$

with expected values

$$\mathbb{E}[R] = \frac{\mathbb{E}[S_T] - S_0}{S_0} \quad \Longrightarrow \quad S_0 = \frac{\mathbb{E}[S_T]}{1 + \mathbb{E}[R]}$$

Using the CAPM model, we have

$$S_0 = \frac{\mathbb{E}[S_T]}{1 + r + \beta(\mu_M - r)} \tag{8.6}$$

Remarks:

1. We need the distribution of S_T to price an asset in this model;
2. The estimation of β becomes a main factor in this pricing model. This is the way to have the input of the market in your pricing of the asset.

Example 3 *(hurdle rate) The hurdle rate of a project/investment is the minimum acceptable investment return to the investors. It is a very useful factor in the decision making process for the investors. Here we consider two projects with forecast payments in the future:*

Project	A	B
Beta	1.5	1.4
Initial Investment	\$10,000	\$10,000
Expected Payoffs	\$8,000 in 2 years \$16,000 in 5 years	\$9,000 in 2 years \$9,000 in 5 years \$9,000 in 8 years

The hurdle rate is calculated as

$$h = \mathbb{E}[R_{project}] = r + \beta_{project}\left(\mathbb{E}[R_M] - r\right)$$

Suppose $r = 2\%$ and $\mathbb{E}[R_M] = 20\%$. We have

$$\mathbb{E}[R_A] = 0.02 + 1.5 \times (0.2 - 0.02) = 0.29$$

$$\mathbb{E}[R_B] = 0.02 + 1.4 \times (0.2 - 0.02) = 0.29$$

When we calculate the net present values, it is important to use the correct discount rate for the purpose. In this case, we are trying to make a decision in comparison with the expected market return. Therefore we have

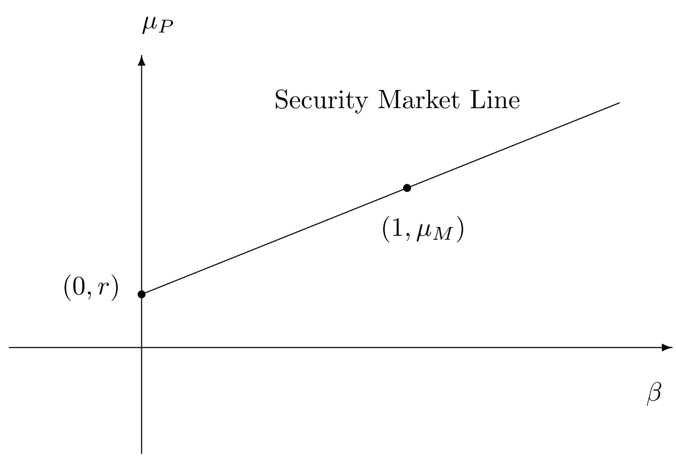
$$NPV_A = -10,000 + \frac{8,000}{1.29^2} + \frac{16,000}{1.29^5} \approx -713 < 0$$

$$NPV_B = -10,000 + 9,000 \left(\frac{1}{1.272^2} + \frac{1}{1.272^5} + \frac{1}{1.272^5} \right) \approx -421 < 0$$

This shows that neither project is worthy of the effort so they should be abandoned. Imagine that a different expected market return and beta can change the picture completely.

8.10 Security Market Line (SML)

The CAPM model plots the expected return of the individual asset as a function of the expected return of the market. Alternatively we can also plot the expected return of the asset as a function of its beta. The CAPM model then gives a straight line with slope $\mu_M - r$. This line can serve as a criterion to determine if the stock price is overvalued or undervalued. Imagine that you obtained information about the expected stock return (μ) and the beta of the stock (as an indicator for market correlation), if the point (β, μ) sits below the SML line, then we can argue that the stock is overvalued, since the expected return is lower than what's expected out of the CAPM model.



We can also use the CAPM model to decompose risk sources for individual assets, which implies

$$\mathbb{E}\left[R_i - r\right] = \beta \mathbb{E}[R_M - r]$$

We can write

$$R_i - r = \beta(R_M - r) + \epsilon_i, \qquad \mathbb{E}[\epsilon_i] = 0$$

If we can model ϵ by a normal random variable, and assume that ϵ_i is independent of the market, taking the variances of both sides gives

$$\sigma_i^2 = \mathrm{Var}(R_i) = \beta_i^2 \sigma_M^2 + \mathrm{Var}(\epsilon_i)$$

The left-hand-side can be seen as the total risk of the stock, and the first term on the right hand side is the market risk, and the second term on the right is what we called the *idiosyncratic* risk of the stock.
Different stocks will have different correlation with the market, and we can estimate their correlation with each other via their correlation with the market:

$$\mathrm{Cov}(R_i, R_j) = \beta_i \beta_j \sigma_M^2$$