To estimate the second term, we use the fact that $\sum b_k$ converges, which implies $\sum_{k>K}\sum_{\ell=1}^{\infty}|a_{k\ell}|<\epsilon/2$ whenever $K>K_0$, for some K_0. For the first term above, note that $\sum_{k\le K}\sum_{\ell>L}|a_{k\ell}|\le\sum_{k=1}^{\infty}\sum_{\ell>L}|a_{k\ell}|$. But the argument above guarantees that we can interchange these last two sums; also $\sum_{\ell=1}^{\infty}\sum_{k=1}^{\infty}|a_{k\ell}|<\infty$, so that for all $L>L_0$ we have $\sum_{\ell>L}\sum_{k=1}^{\infty}|a_{k\ell}|<\epsilon/2$. Taking $N>\max(L_0,K_0)$ completes the proof of (ii).

The proof of (iii) is a direct consequence of (ii). Indeed, given any rectangle

$$R(K,L)=\{(k,\ell)\in\mathbb{N}\times\mathbb{N}:1\le k\le K\text{ and }1\le\ell\le L\},$$

there exists M such that the image of $[1,M]$ under the map $m\mapsto(k(m),\ell(m))$ contains $R(K,L)$.

When U denotes any open set in $\mathbb{R}^2$ that contains the origin, we define for $R>0$ its dilate $U(R)=\{y\in\mathbb{R}^2: y=Rx \text{ for some } x\in U\}$, and we can apply (ii) to see that

$$A=\lim_{R\to\infty}\sum_{(k,\ell)\in U(R)}a_{k\ell}.$$

In other words, under condition (8) the double sum $\sum_{k\ell}a_{k\ell}$ can be evaluated by summing over discs, squares, rectangles, ellipses, etc.

Finally, we leave the reader with the instructive task of finding a sequence of complex numbers $\{a_{k\ell}\}$ such that

$$\sum_k\sum_\ell a_{k\ell}\neq\sum_\ell\sum_k a_{k\ell}.$$

[Hint: Consider $\{a_{k\ell}\}$ as the entries of an infinite matrix with 0 above the diagonal, -1 on the diagonal, and $a_{k\ell}=2^{\ell-k}$ if $k>\ell$.]

3 Exercises

1. Suppose that $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers such that the partial sums

$$A_n=a_1+\cdots+a_n$$

are bounded. Prove that the Dirichlet series

$$\sum_{n=1}^{\infty}\frac{a_n}{n^s}$$

converges for $\mathrm{Re}(s)>0$ and defines a holomorphic function in this half-plane.

[Hint: Use summation by parts to compare the original (non-absolutely convergent) series to the (absolutely convergent) series $\sum A_n(n^{-s}-(n+1)^{-s})$. An estimate for the term in parentheses is provided by the mean value theorem. To prove that the series is analytic, show that the partial sums converge uniformly on every compact subset of the half-plane $\mathrm{Re}(s)>0$.]