MATH 3210 Spring 2024

Fourth Midterm Exam

Professor: Y.P. Lee

Solution

INSTRUCTION: Show all of your work. Make sure your answers are clear and legible. Use *specified* method to solve the question. It is not necessary to simplify your final answers.

Problem 1	20	________
Problem 2	20	________
Problem 3	20	________
Problem 4	20	________
Problem 5	20	________
Total	100	________

PROBLEM 1

(a) Find the Taylor series for $\dfrac{1}{\sqrt{1-x}}$ valid on $(-1,1)$ with center $a=0$. (No need to show $\lim R_n(x)=0$.) (15 pt)
(b) Use (a) to get power series expansion for $\dfrac{1}{\sqrt{1-u^2}}$ on $(-1,1)$. (5 pt)

Solution. (HW §6.5 #8)
(a) Let $f(x)=(1-x)^{-1/2}$. Then

$$f^{(n)}(x)=(\frac{1}{2})(\frac{3}{2})\cdots(\frac{2n-1}{2})(1-x)^{-(2n+1)/2}.$$

Therefore,

$$\frac{f^{(n)}(0)}{n!}=\frac{(2n)!}{n!n!2^{2n}}$$

and the Taylor series at $x=0$ is

$$f(x)=(1-x)^{-1/2}=\sum_{n=0}^{\infty}\frac{(2n)!}{n!n!2^{2n}}x^n.$$

(b) Substitute $x=u^2$ into the above formula, we have

$$(1-u^2)^{-1/2}=\sum_{n=0}^{\infty}\frac{(2n)!}{n!n!2^{2n}}u^{2n}.$$

□

PROBLEM 2

(a) What is Taylor's formula (with the remainder term) for $\ln(x)$ with a=1?
(b) Find a power series in x centered at 1 which converges to $\ln(x)$. What is the radius of convergence? (No need to show $\lim R_n(x) = 0$.)

Solution. (HW §6.5 # 7)

(a) Let $f(x) = \ln x$. Then for $n \geq 1$,

$$f^{(k)}(x) = (k-1)!(-1)^{k-1}x^{-k},$$
$$\frac{f^{(k)}(1)}{k!} = \frac{(-1)^{k-1}}{k}$$

We note that $\ln(1) = 0$. Therefore,

$$\ln(x) = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}(x-1)^k + R_n(x),$$
$$R_n(x) = \frac{(-1)^n}{c^{n+1}(n+1)}(x-1)^{n+1},$$

for some c between 1 and x.

(b)

$$\ln(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}(x-1)^k.$$

The radius of convergence is

$$R = \frac{1}{\limsup |1/k|^{1/k}} = \lim |k|^{1/k} = 1.$$

□

PROBLEM 3

Prove the function $f(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^2}$ is continuous on the interval $[-1, 1]$.

Solution. (HW §6.4 # 1)

Apply Weierstress M-test. For $x \in [-1, 1]$, we have

$$|\frac{x^k}{k^2}| \leq \frac{1}{k^2}.$$

Since $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, we conclude that $f(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^2}$ converges uniformly on $[-1, 1]$. Since each $\frac{x^k}{k^2}$ is continuous, by Theorem 6.4.2 $f(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^2}$ is continuous. □

PROBLEM 4

In the following two cases, determine whether the given series converges absolutely, converges conditionally, or diverges. Justify your answer.

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k-1}2^k}{2^k+k^2}$. (b) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^{2+(-1)^k}}$.

Solution. (HW §6.3 # 4, 5)

(a) The series diverges by the Term Test, as

$$\lim_{k\to\infty} \frac{2^k}{2^k+k^2} = 1$$

by L'Hôpital's rule.

(b) The series diverges. We know that the summation of the even terms

$$\sum_{l=1}^{\infty} \frac{(-1)}{(2l)^3} = \frac{-1}{8}\sum_{l=1}^{\infty} \frac{1}{l^3}$$

converges. Let the limit be denoted by $-K$, with $K > 0$. Let the partial sum

$$s_n = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k^{2+(-1)^k}},$$

then it is easy to see that

$$s_{2n} = \sum_{l=1}^{n} \frac{1}{2l-1} + \sum_{l=1}^{n} \frac{(-1)}{(2l)^3} > \sum_{l=1}^{n} \frac{1}{2l-1} - K.$$

Since the first term diverges, the entire series must also diverge.

We note that the Alternating Series Test does not apply as the series is alternating, but *not* non-decreasing. □

PROBLEM 5

(a) Why is the integral $\int_0^1 (\ln x)\, dx$ an *improper integral*? (5 pt)

(b) Does the above improper integral converge? If so, what does it converge to? (15 pt)

Solution. (HW §5.4 #12)

(a) The integrant has a singularity at $x = 0$,

$$\lim_{x \to 0^+} \ln x = -\infty.$$

It is therefore an improper integral and

$$\int_0^1 (\ln x)\, dx = \lim_{a \to 0^+} \int_a^1 \ln x dx.$$

(b) We apply integration by parts

$$\begin{aligned} \int_0^1 (\ln x)\, dx &= \lim_{a \to 0^+} \int_a^1 \ln x dx \\ &= \lim_{a \to 0^+} \left([x \ln x]_a^1 - \int_a^1 x \cdot x^{-1}\, dx \right) \\ &= \lim_{a \to 0^+} \left([0 - a \ln a] - [1 - a] \right) = -1, \end{aligned}$$

where the last equality uses L'Hôpital rule.

Hence, the improper integral converges to -1. □