MATH 3210 Spring 2024

Third Midterm Exam

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Solution

INSTRUCTION: Show all of your work. Make sure your answers are clear and legible. Use *specified* method to solve the question. It is not necessary to simplify your final answers.

Problem 1	20	______
Problem 2	20	______
Problem 3	20	______
Problem 4	20	______
Problem 5	20	______
Total	100	______

PROBLEM 1

Prove that the function $f(x) = x \sin \frac{1}{x}$ is uniformly continuous on $(0, 1)$.

Solution. (HW §3.3 #10 and §3.1 #12)

1. We show that f is continuous at 0, if we define $f(0) = 0$.
$\forall \varepsilon > 0$, let $\delta = \varepsilon > 0$. Then $\forall x, |x - 0| < \delta$, we have

$$|f(x) - f(0)| = |x \sin \frac{1}{x} - 0| = |x||\sin \frac{1}{x}| \leq |x| < \delta = \varepsilon.$$

2. We show that f is continuous at $a \in (0, 1]$.

In fact, away from 0, $f(x)$ is differentiable and hence continuous (Theorem 4.2.5). Alternatively, this can be proven by $\varepsilon - \delta$ definition.

3. Therefore, f is continuous on a closed bounded interval $[0, 1]$ and hence uniformly continuous on $[0, 1]$ by Theorem 3.3.4. This implies that f is uniformly continuous on $(0, 1)$. □

PROBLEM 2

Use the Intermediate Value Theorem to prove that, if n is a natural number, then every positive number a has a unique positive nth root.

Solution. (HW §3.2 #10)

Let $f(x) = x^n$. It is a smooth (and hence continuous) function on $\mathbb{R}$. Since a is a positive number, it is easy to see that

$$f(0) = 0 < a < f(a+1) = (a+1)^n.$$

By the Intermediate Value Theorem, $\exists c \in [0, a+1]$ such that $f(c) = c^n = a$. Furthermore, $f(0) \neq a$ and $f(a+1) \neq a$, $c \in (0, a+1)$.

Since $f'(x) > 0$ for $x > 0$, f is a strictly increasing function. Therefore, the above c must be unique. Indeed, $c = (a)^{1/n}$ as $f(c) = c^n = a$. $\square$

PROBLEM 3

Prove that the sequence $\{\frac{x}{n}\}$ converges uniformly to 0 on each bounded interval, but does not converge uniformly on $\mathbb{R}$.

Solution. (HW §3.4 #1)

(1) Let I be a given bounded interval. Therefore, $\exists M > 0$ such that $\forall x \in I, |x| < M$. (More concretely, $I = [a, b], (a, b), (a, b], [a, b)$ be a bounded interval. Let $0 < M \in \mathbb{R}$ such that $|a| < M$ and $|b| < M$.)

$\forall \varepsilon > 0$, let $N = \frac{M}{\varepsilon}$. Then $\forall x \in I$, $\forall n > N$, we have

$$|\frac{x}{n} - 0| = \frac{|x|}{n} < \frac{M}{N} = \varepsilon.$$

This show that $\{\frac{x}{n}\}$ converges uniformly to 0 on I.

(2) Not converge uniformly: $\exists \varepsilon > 0$, such that $\forall N, \exists x, \exists n > N$ with $|\frac{x}{n} - 0| \geq \varepsilon$.

Choose $\varepsilon = 1$. For any N, let $(n, x) = (n, 2n)$, where $n > N$ (e.g., $n = N + 1$). Then $|\frac{x}{n} - 0| = 2 > 1 = \varepsilon$. $\square$

PROBLEM 4

Prove that if f is a differentiable function on $(0, \infty)$ and f and f' both have finite limits at ∞, then $\lim_{x\to\infty} f'(x) = 0$.

Solution. (HW §4.3 #5)

Apply MVT to $[a, a+1]$. Then, $\exists c_a \in [a, a+1]$ such that

$$f'(c_a) = f(a+1) - f(a). \tag{1}$$

Note that c_a depends on a. Then it is easy to see that as $a \to \infty$, c_a goes to ∞ and the RHS of (1) goes to 0. This implies that

$$\lim_{x\to\infty} f'(x) = 0.$$

Alternatively,

(1) Let $\lim_{x\to\infty} f(x) = L \in \mathbb{R}$. This by definition means that $\forall \varepsilon > 0$, $\exists N > 0$, such that $\forall x > N$ we have $|f(x) - L| < \varepsilon$.

(2) Let $\lim_{x\to\infty} f'(x) = L' \in \mathbb{R}$. This by definition means that $\forall \varepsilon' > 0$, $\exists N' > 0$, such that $\forall x > N'$ we have $|f'(x) - L'| < \varepsilon'$.

If $\mathbf{L' \neq 0}$, let $\varepsilon = \frac{L'}{4}$. By (1), we have a corresponding N. Let $\varepsilon' = \varepsilon$ and we have a corresponding N' by (2).

Therefore, $\forall x > \text{Max}(N, N')$ we have

$$|f(x) - f(x+1)| \leq |f(x) - L| + |f(x+1) - L| < 2\varepsilon = \frac{L'}{2}. \tag{2}$$

By the Mean Value Theorem, we have $c \in (x, x+1)$ such that

$$f'(c) = \frac{f(x+1) - f(x)}{(x+1) - x} = f(x+1) - f(x).$$

By (2), we have

$$|f'(c)| = |f(x+1) - f(x)| < \frac{L'}{2}. \tag{3}$$

However, $c > x > N'$, therefore by (2) we have

$$|f'(c) - L'| < \varepsilon' = \frac{L'}{4}. \tag{4}$$

Equations (3) and (4) give a contradiction. Hence $L' = 0$. □

PROBLEM 5

Find $\lim_{x\to 0^+} x^x$.

Solution. (HW §4.4 #10)

$$\lim_{x\to 0} x^x = \lim_{x\to 0} e^{x\ln x} = e^{\lim_{x\to 0} \frac{\ln x}{1/x}} = e^{\lim_{x\to 0} \frac{1/x}{-1/x^2}} = e^{\lim_{x\to 0} -x} = e^0 = 1,$$

where Theorem 4.1.12 and the L'Hôpital's Rule are used for the third and the fourth =. □