MATH 3210 Spring 2024

Second Midterm Exam

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Solution

Problem 1	30	______
Problem 2	20	______
Problem 3	20	______
Problem 4	10	______
Problem 5	20	______
Total	100	______

PROBLEM 1

(a) What is a *Dedekind cut*?
(b)How do you identify a Dedekind cut with a real number?

Solution. (a) A subset $L \subset \mathbb{Q}$ is called a Dedekind cut if it satisfies the following three conditions:

(A) $L \neq \emptyset$ and $L \neq \mathbb{Q}$,
(B) L has no largest element,
(C) if $x \in L$, then so is every $y \in \mathbb{Q}$ with $y < x$.

(b) Let $a \in \mathbb{R}$, then the corresponding Dedekind cut is

$$L_a := \{x \in \mathbb{Q} : x < a\}.$$

□

PROBLEM 2

Make an educated guess of the limit and then prove your guess of

$$\lim_{n\to\infty} \frac{2n^2+1}{n^2+1}$$

by the **definition**.

Solution.

$$\lim_{n\to\infty} \frac{2n^2+1}{n^2+1} = 2$$

Proof. $\forall \varepsilon > 0$, let $N > 0$ be any real number such that

$$|\frac{1}{N^2+1}| < \varepsilon.$$

Then $\forall n > N$, we have

$$|\frac{2n^2+1}{n^2+1} - 2| = |\frac{1}{n^2+1}| < |\frac{1}{N^2+1}| < \varepsilon$$

$\square$

$\square$

PROBLEM 3

Let $\{a_n\}$ be a sequence of real numbers such that $\lim a_n = 0$, and let $\{b_n\}$ be a bounded sequence. Prove that $\lim a_n b_n = 0$ **by definition**.

Solution. $\{b_n\}$ bounded implies that $\exists M > 0$ such that $|b_n| < M$.

$\lim a_n = 0 \Leftrightarrow \forall \varepsilon_a, \exists N_a$, such that $\forall n > N_a$, we have $|a_n - 0| < \varepsilon_a$.

Now we show $\lim a_n b_n = 0$. $\forall \varepsilon$, choose $\varepsilon_a = \varepsilon / M$ and get the corresponding N_a. Let $N = N_a$. Then $\forall n > N$, we have

$$|a_n b_n - 0| < |a_n| M < \varepsilon_a M = \varepsilon.$$

$\square$

PROBLEM 4

Let $\{s_n\}$ be the sequence of partial sums of a series with non-negative terms. That is,

$$s_n = \sum_{k=1}^{n} a_k, \quad a_k \geq 0, \forall k \in \mathbb{N}.$$

Prove that $\lim s_n$ exists.

Solution. $\{s_n\}$ is a non-decreasing sequence. By Theorem 2.4.6, every monotone sequence has a limit. $\square$

PROBLEM 5

Prove that a sequence which satisfies $|a_{n+1} - a_n| < 2^{-n}$ for all n is a Cauchy sequence.

Solution. $\forall\varepsilon$, let $N \in \mathbb{R}$ satisfying

$$2^{-N+1} < \varepsilon.$$

Without loss of generality, assume $m > n$. Then $\forall m, n > N$, we have

$$|a_m - a_n| < |a_m - a_{m-1}| + \cdots + |a_{n+1} - a_n| < 2^{-m+1} + \cdots + 2^{-n} < 2^{-n+1} < 2^{-N+1} < \varepsilon.$$

Therefore, $\{a_n\}$ is a Cauchy sequence by definition.

Alternatively, one can try to show that $\lim a_n$ exists as a real number, and use Theorem 2.5.8. $\square$