MATH 3210 Spring 2024

First Midterm Exam

Professor: Y.P. Lee

LAST NAME Solutions

First NAME ________

Section ________

ID Number ________

INSTRUCTION: Show all of your work. Make sure your answers are clear and legible. Use *specified* method to solve the question. It is not necessary to simplify your final answers.

Problem 1	20	____
Problem 2	20	____
Problem 3	20	____
Problem 4	20	____
Problem 5	20	____
Total	100	____

PROBLEM 1

If $f : A \to B$ is a one-to-one function and E, F are subsets of A, show that

$$f(E \cap F) = f(E) \cap f(F)$$

(20 pt)

pf $(\subset)$ $b \in f(E \cap F)$

$\Rightarrow \exists a \in E \cap F$ s.t. $f(a) = b$

$\Rightarrow b \in f(E)$ and $b \in f(F)$

$\Rightarrow b \in f(E) \cap f(F)$

$(\supset)$ $b \in f(E) \cap f(F)$

$\Rightarrow \exists e \in E$ and $\exists a \in F$ s.t. $b = f(e)$ and $b = f(a)$

since $f = 1\text{-}1$ $\Rightarrow e = a \in E \cap F$

$\Rightarrow b \in f(E \cap F)$ ¤

PROBLEM 2

Fix $m \in \mathbb{N}$. Use Peano's axioms and their immediate consequences to *define* a sequence $\{m+n\}_{n\in\mathbb{N}}$ inductively. (20 pt)

Using Thm 1.2.3. can define $\{m+n\}_{n\in\mathbb{N}}$ inductively as

$$\begin{cases} m+1 = S(m) \\ m+S(n) = S(m+n) \end{cases}$$

$$\left[\begin{array}{l} \text{i.e., } x_1 = S(m) \\ \quad f_n(x) = x+1 \ (= S(x)) \end{array} \right]$$

PROBLEM 3

Prove that $1 + n = n + 1$ for all $n \in \mathbb{N}$. (20 pt) assuming associativity of additions.

Pf: By math induction

P_n: $1 + n = n + 1$

• $\boxed{n=1}$ $1 + 1 = 1 + 1$ ✓

• Assume P_n,

$1 + (n+1) = (1+n) + 1$ by associativity of addition (Example 1.2.5)

$= (n+1) + 1$ by P_n

$\Rightarrow 1 + (n+1) = (n+1) + 1 \Leftrightarrow P_{n+1}$

By math ind. $\{P_n\}_{n \in \mathbb{N}}$ proven #

PROBLEM 4

Prove the commutative law for addition, i.e., $n+m=m+n$, holds for all $m, n \in \mathbb{N}$. (20 pt)

Pf. Fix an arbitrary $m \in \mathbb{N}$. Induction on n

P_n : $n+m=$ ~~$n+m$~~ $m+n$

- $n=1$ $\quad 1+m=m+1$ holds by Problem 3.
- Assume P_n holds $\quad n+m=m+n$

~~$s(n)+m$ $(n+1)+m=$~~

$(+1)$ to the right $\quad (n+m)+1=(m+n)+1$

by associativity $\quad n+(m+1)=m+(n+1)$

$\quad\quad ||$

by Prob 3 $\quad n+(1+m)=m+(n+1)$

by associativity $\quad (n+1)+m = m+(n+1) \iff P_{n+1}$

By math ind. $\{P_n\}_{n\in\mathbb{N}}$ proven #

PROBLEM 5

Prove that every rational solution of a polynomial equation

$$x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = 0$$

with integer coefficients a_k, is an integer solution. (20 pt)

Pf Let $x = \frac{p}{q} \in \mathbb{Q}$ be a solution, (WLOG)

s.t. p, q have no common factor.

$$\left(\frac{p}{q}\right)^n + a_{n-1}\left(\frac{p}{q}\right)^{n-1} + \cdots + a_1\left(\frac{p}{q}\right) + a_0 = 0$$

multiplying both sides by q^n:

$$p^n + a_{n-1}p^{n-1}q + \cdots + a_1 p q^{n-1} + a_0 q^n = 0$$

$$p^n + a_{n-1}p^{n-1}q + \cdots + a_1 p q^{n-1} = -a_0 q^n$$

Since q divides RHS, it must divide LHS.

$$\Rightarrow q \mid p^n + \underbrace{a_{n-1}p^{n-1}q + \cdots + a_1 p q^{n-1}}_{= q(a_{n-1}p^{n-1} + \cdots + a_1 p q^{n-2})}$$

$$\Rightarrow q \mid p^n$$

Since q and p have no common factor $\Rightarrow q \nmid p^n \ \forall n$

$\boxed{\text{unless } q = 1}$ $\Rightarrow x = \frac{p}{1} \in \mathbb{Z}$ #