Chapter 1

Introduction

1.1 PDE motivations and context

The aim of this is to introduce and motivate partial differential equations (PDE). The section also places the scope of studies in APM346 within the vast universe of mathematics. A partial differential equation (PDE) is an equation involving partial derivatives. This is not so informative so let’s break it down a bit.

1.1.1 What is a differential equation?

An ordinary differential equation (ODE) is an equation for a function which depends on one independent variable which involves the independent variable, the function, and derivatives of the function:

\[ F(t, u(t), u'(t), u''(t), u'''(t), \ldots, u^{(m)}(t)) = 0. \]

This is an example of an ODE of order \( m \) where \( m \) is a highest order of the derivative in the equation. Solving an equation like this on an interval \( t \in [0, T] \) would mean finding a function \( t \mapsto u(t) \in \mathbb{R} \) with the property that \( u \) and its derivatives satisfy this equation for all values \( t \in [0, T] \).

The problem can be enlarged by replacing the real-valued \( u \) by a vector-valued one \( \mathbf{u}(t) = (u_1(t), u_2(t), \ldots, u_N(t)) \). In this case we usually talk about system of ODEs.

Even in this situation, the challenge is to find functions depending upon exactly one variable which, together with their derivatives, satisfy the equation.
Chapter 1. Introduction

What is a partial derivative?
When you have function that depends upon several variables, you can differentiate with respect to either variable while holding the other variable constant. This spawns the idea of partial derivatives. As an example, consider a function depending upon two real variables taking values in the reals:

\[ u : \mathbb{R}^n \rightarrow \mathbb{R}. \]

As \( n = 2 \) we sometimes visualize a function like this by considering its graph viewed as a surface in \( \mathbb{R}^3 \) given by the collection of points

\[ \{(x, y, z) \in \mathbb{R}^3 : z = u(x, y)\}. \]

We can calculate the derivative with respect to \( x \) while holding \( y \) fixed. This leads to \( u_x \), also expressed as \( \partial_x u \), \( \frac{\partial u}{\partial x} \), and \( \frac{\partial}{\partial x} u \). Similarly, we can hold \( x \) fixed and differentiate with respect to \( y \).

What is PDE?
A partial differential equation is an equation for a function which depends on more than one independent variable which involves the independent variables, the function, and partial derivatives of the function:

\[ F(x, y, u(x, y), u_x(x, y), u_y(x, y), u_{xx}(x, y), u_{xy}(x, y), u_{yx}(x, y), u_{yy}(x, y)) = 0. \]

This is an example of a PDE of order 2. Solving an equation like this would mean finding a function \( (x, y) \rightarrow u(x, y) \) with the property that \( u \) and its derivatives satisfy this equation for all admissible arguments.

Similarly to ODE case this problem can be enlarged by replacing the real-valued \( u \) by a vector-valued one \( u(t) = (u_1(x, y), u_2(x, y), \ldots, u_N(x, y)) \). In this case we usually talk about system of PDEs.

Where PDEs are coming from?
PDEs are often referred as Equations of Mathematical Physics (or Mathematical Physics but it is incorrect as Mathematical Physics is now a separate field of mathematics) because many of PDEs are coming from different
domains of physics (acoustics, optics, elasticity, hydro and aerodynamics, electromagnetism, quantum mechanics, seismology etc).

However PDEs appear in other fields of science as well (like quantum chemistry, chemical kinetics); some PDEs are coming from economics and financial mathematics, or computer science.

Many PDEs are originated in other fields of mathematics.

Examples of PDEs
(Some are actually systems)

Simplest first order equation

$$u_x = 0.$$  

Transport equation

$$u_t + cu_x = 0.$$  

\(\bar{\partial}\) equation

$$\bar{\partial} f := \frac{1}{2} (f_x + if_y) = 0,$$

(\(\bar{\partial}\) is known as “di-bar” or Wirtinger derivatives), or as \(f = u + iv\)

$$\begin{cases} u_x - v_y = 0, \\ u_y + v_x = 0. \end{cases}$$

Those who took Complex variables know that those are Cauchy-Riemann equations.

Laplace’s equation (in 2D)

$$\Delta u := u_{xx} + u_{yy} = 0$$

or similarly in the higher dimensions.
Chapter 1. Introduction

Heat equation

\[ u_t = k \Delta u; \]

(The expression \( \Delta \) is called the Laplacian (Laplace operator) and is defined as \( \partial_x^2 + \partial_y^2 + \partial_z^2 \) on \( \mathbb{R}^3 \)).

Schrödinger equation (quantum mechanics)

\[ i\hbar \partial_t \psi = \left( -\frac{\hbar^2}{2m} \Delta + V \right) \psi. \]

Here \( \psi \) is a complex-valued function.

Wave equation

\[ u_{tt} - c^2 \Delta u = 0; \]

sometimes \( \Box := c^{-2} \partial_t^2 - \Delta \) is called (d’Alembertian or d’Alembert operator).

It appears in elasticity, acoustics, electromagnetism and so on.

One-dimensional wave equation

\[ u_{tt} - c^2 u_{xx} = 0 \]

often is called string equation and describes f.e. a vibrating string.

Oscillating rod or plate (elasticity) Equation of vibrating rod (with one spatial variable)

\[ u_{tt} + Ku_{xxxx} = 0 \]

or vibrating plate (with two spatial variables)

\[ u_{tt} + K \Delta^2 u = 0. \]

Maxwell equations (electromagnetism) in vacuum

\[
\begin{align*}
E_t - c \nabla \times H &= 0, \\
H_t + c \nabla \times E &= 0, \\
\nabla \cdot E &= \nabla \cdot H = 0.
\end{align*}
\]

Here \( E \) and \( H \) are vectors of electric and magnetic intensities, so the first two lines are actually \( 6 \times 6 \) system. The third line means two more equations, and we have \( 8 \times 6 \) system. Such systems are called overdetermined.
Chapter 1. Introduction

Dirac equations (relativistic quantum mechanics)

\[ i\hbar \partial_t \psi = (\beta mc^2 - \sum_{1 \leq k \leq 3} i\hbar \alpha_k \partial_{x_k}) \psi \]

with Dirac $4 \times 4$-matrices $\alpha_1, \alpha_2, \alpha_3, \beta$. Here $\psi$ is a complex 4-vector, so in fact we have $4 \times 4$ system.

Elasticity equations (homogeneous and isotropic)

\[
\mathbf{u}_{tt} = \lambda \Delta \mathbf{u} + \mu \nabla (\nabla \cdot \mathbf{u}).
\]

homogeneous means “the same in all places” (an opposite is called inhomogeneous) and isotropic means “the same in all directions” (an opposite is called anisotropic).

Navier-Stokes equation (hydrodynamics for incompressible liquid)

\[
\begin{cases}
\rho \mathbf{v}_t + (\mathbf{v} \cdot \nabla) \rho \mathbf{v} - \nu \Delta \mathbf{v} = -\nabla p, \\
\nabla \cdot \mathbf{v} = 0,
\end{cases}
\]

where $\rho$ is a (constant) density, $\mathbf{v}$ is a velocity and $p$ is the pressure; when viscosity $\nu = 0$ we get Euler equation

\[
\begin{cases}
\rho \mathbf{v}_t + (\mathbf{v} \cdot \nabla) \rho \mathbf{v} = -\nabla p, \\
\nabla \cdot \mathbf{v} = 0.
\end{cases}
\]

Both of them are $4 \times 4$ systems.

Yang-Mills equation (elementary particles theory)

\[ \partial_{x_j} F_{jk} + [A_j, F_{jk}] = 0, \]

\[ F_{jk} := \partial_{x_j} A_k - \partial_{x_k} A_j + [A_j, A_k], \]

where $A_k$ are traceless skew-Hermitian matrices. Their matrix elements are unknown functions. This is a 2nd order system.
Einstein equation for general relativity

\[ G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}, \]

where \( G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} \) is the Einstein tensor, \( g_{\mu\nu} \) is the metric tensor (unknown functions), \( R_{\mu\nu} \) is the Ricci curvature tensor, and \( R \) is the scalar curvature, \( T_{\mu\nu} \) is the stress–energy tensor, \( \Lambda \) is the cosmological constant and \( \kappa \) is the Einstein gravitational constant. Components of Ricci curvature tensor are expressed through the components of the metric tensor, their first and second derivatives. This is a 2nd order system.

Black-Scholes equation  

Black-Scholes Equation (Financial mathematics) is a partial differential equation (PDE) governing the price evolution of a European call or European put under the Black–Scholes model. Broadly speaking, the term may refer to a similar PDE that can be derived for a variety of options, or more generally, derivatives:

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + r S \frac{\partial V}{\partial S} - r V = 0
\]

where \( V \) is the price of the option as a function of stock price \( S \) and time \( t \), \( r \) is the risk-free interest rate, and \( \sigma \) is the volatility of the stock.

Do not ask me what this means!

Remark 1.1.1. (a) Some of these examples are actually not single PDEs but the systems of PDEs.

(b) In all these examples there are spatial variables \( x, y, z \) and often time variable \( t \) but it is not necessarily so in all PDEs. Equations, not including time, are called stationary (an opposite is called nonstationary).

(c) Equations could be of different order with respect to different variables and it is important. However if not specified the order of equation is the highest order of the derivatives invoked.

(d) In this class we will deal mainly with the wave equation, heat equation and Laplace equation in their simplest forms.
1.2 Initial and boundary value problems

1.2.1 Problems for PDEs

We know that solutions of ODEs typically depend on one or several constants. For PDEs situation is more complicated. Consider simplest equations

\begin{align*}
  u_x &= 0, \quad (1.2.1) \\
  v_{xx} &= 0, \quad (1.2.2) \\
  w_{xy} &= 0 \quad (1.2.3)
\end{align*}

with \( u = u(x, y) \), \( v = v(x, y) \) and \( w = w(x, y) \). Equation (1.2.1) could be treded as an ODE with respect to \( x \) and its solution is a constant but this is not a genuine constant as it is constant only with respect to \( x \) and can depend on other variables; so \( u(x, y) = \phi(y) \).

Meanwhile, for solution of (1.2.2) we have \( v_x = \phi(y) \) where \( \phi \) is an arbitrary function of one variable and it could be considered as ODE with respect to \( x \) again; then \( v(x, y) = \phi(y)x + \psi(y) \) where \( \psi(y) \) is another arbitrary function of one variable.

Finally, for solution of (1.2.3) we have \( w_y = \phi(y) \) where \( \phi \) is an arbitrary function of one variable and it could be considered as ODE with respect to \( y \); then \( (w - g(y))_y = 0 \) where \( g(y) = \int \phi(y) \, dy \), and therefore \( w - g(y) = f(x) \implies w(x, y) = f(x) + g(y) \) where \( f, g \) are arbitrary functions of one variable.

Considering these equations again but assuming that \( u = u(x, y, z) \), \( v = v(x, y, z) \) we arrive to \( u = \phi(y, z) \), \( v = \phi(y, z)x + \psi(y, z) \) and \( w = f(x, z) + g(y, z) \) where \( f, g \) are arbitrary functions of two variables.

Solutions to PDEs typically depend not on several arbitrary constants but on one or several arbitrary functions of \( n - 1 \) variables. For more complicated equations this dependence could be much more complicated and implicit. To select a right solutions we need to use some extra conditions.

The sets of such conditions are called Problems. Typical problems are

(a) IVP (Initial Value Problem): one of variables is interpreted as time \( t \) and conditions are imposed at some moment; f.e. \( u|_{t=t_0} = u_0 \);

(b) BVP (Boundary Value Problem) conditions are imposed on the boundary of the spatial domain \( \Omega \); f.e. \( u|_{\partial \Omega} = \phi \) where \( \partial \Omega \) is a boundary of \( \Omega \) and \( \phi \) is defined on \( \partial \Omega \);
Chapter 1. Introduction

(c) IVBP (Initial-Boundary Value Problem a.k.a. mixed problem): one of
variables is interpreted as time $t$ and some conditions are imposed at
some moment but other conditions are imposed on the boundary of
the spatial domain.

Remark 1.2.1. In the course of ODEs students usually consider IVP only.
F.e. for the second-order equation like

$$u_{xx} + a_1 u_x + a_2 u = f(x)$$

such problem is $u|_{x=x_0} = u_0$, $u_x|_{x=x_0} = u_1$. However one could consider
BVPs like

$$(\alpha_1 u_x + \beta_1 u)|_{x=x_1} = \phi_1,$$
$$(\alpha_2 u_x + \beta_2 u)|_{x=x_2} = \phi_2,$$

where solutions are sought on the interval $[x_1, x_2]$. Such are covered in
advanced chapters of some of ODE textbooks (but not covered by a typical
ODE class). We will need to cover such problems later in this class.

1.2.2 Notion of “well-posedness”

We want that our PDE (or the system of PDEs) together with all these
conditions satisfied the following requirements:

(a) Solutions must exist for all right-hand expressions (in equations and
conditions)–existence;

(b) Solution must be unique–uniqueness;

(c) Solution must depend on these right-hand expressions continuously,
which means that small changes in the right-hand expressions lead to
small changes in the solution.

Such problems are called well-posed. PDEs are usually studied together
with the problems which are well-posed for these PDEs. Different types of
PDEs admit different problems.

Sometimes however one needs to consider ill-posed problems. In particular, inverse problems of PDEs are almost always ill-posed.
1.3 Classification of equations

1.3.1 Linear and nonlinear equations

Equations of the form

\[ Lu = f(x) \]  \hspace{1cm} (1.3.1)

where \( Lu \) is a partial differential expression linear with respect to unknown function \( u \) is called linear equation (or linear system). This equation is linear homogeneous equation if \( f = 0 \) and linear inhomogeneous equation otherwise. For example,

\[ Lu := a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + a_1u_x + a_2u_y + au = f(x) \]  \hspace{1cm} (1.3.2)

is linear; if all coefficients \( a_{jk}, a_j, a \) are constant, we call it linear equation with constant coefficients; otherwise we talk about variable coefficients.

Otherwise equation is called nonlinear. However there is a more subtle classification of such equations. Equations of the type (1.3.1), where the right-hand expression \( f \) depends on the solution and its lower-order derivatives, are called semilinear, equations where both coefficients and right-hand expression depend on the solution and its lower-order derivatives are called quasilinear. For example,

\[ Lu := a_{11}(x, y)u_{xx} + 2a_{12}(x, y)u_{xy} + a_{22}(x, y)u_{yy} = f(x, y, u, u_x, u_y) \]  \hspace{1cm} (1.3.3)

is semilinear, and

\[ Lu := a_{11}(x, y, u, u_x, u_y)u_{xx} + 2a_{12}(x, y, u, u_x, u_y)u_{xy} + a_{22}(x, y, u, u_x, u_y)u_{yy} = f(x, y, u, u_x, u_y) \]  \hspace{1cm} (1.3.4)

is quasilinear, while

\[ F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yx}) = 0 \]  \hspace{1cm} (1.3.5)

is general nonlinear.

1.3.2 Elliptic, hyperbolic and parabolic equations

General  Consider second order equation (1.3.2):

\[ Lu := \sum_{1 \leq i,j \leq n} a_{ij}u_{x_ix_j} + \text{l.o.t.} = f(x) \]  \hspace{1cm} (1.3.6)
where l.o.t. means lower order terms (that is, terms with \( u \) and its lower order derivatives) with \( a_{ij} = a_{ji} \). Let us change variables \( \mathbf{x} = \mathbf{x}(\mathbf{x}') \). Then the matrix of principal coefficients

\[
A = \begin{pmatrix}
a_{11} & \cdots & a_{1n} \\
\vdots & \ddots & \vdots \\
a_{n1} & \cdots & a_{nn}
\end{pmatrix}
\]

in the new coordinate system becomes \( A' = Q^*AQ \) where \( Q = T^{*-1} \) and \( T = \left( \frac{\partial x_i}{\partial x'_j} \right)_{i,j=1,...,n} \) is a Jacobi matrix. The proof easily follows from the chain rule (Calculus II).

Therefore if the principal coefficients are real and constant, by a linear change of variables matrix of the principal coefficients could be reduced to the diagonal form, where diagonal elements could be either 1, or \(-1\) or 0. Multiplying equation by \(-1\) if needed we can assume that there are at least as many 1 as \(-1\). In particular, for \( n = 2 \) the principal part becomes either \( u_{xx} + u_{yy} \), or \( u_{xx} - u_{yy} \), or \( u_{xx} \) and such equations are called elliptic, hyperbolic, and parabolic respectively (there will be always second derivative since otherwise it would be the first order equation).

This terminology comes from the curves of the second order conical sections: if \( a_{11}a_{22} - a_{12}^2 > 0 \) equation

\[
a_{11}\xi^2 + 2a_{12}\xi\eta + a_{22}\eta^2 + a_1\xi + a_2\eta = c
\]

generically defines an ellipse, if \( a_{11}a_{22} - a_{12}^2 < 0 \) this equation generically defines a hyperbole and if \( a_{11}a_{22} - a_{12}^2 = 0 \) it defines a parabola.

Let us consider equations in different dimensions:

**2D** If we consider only 2-nd order equations with constant real coefficients then in appropriate coordinates they will look like either

\[
u_{xx} + u_{yy} + \text{l.o.t} = f \quad \text{(1.3.7)}
\]

or

\[
u_{xx} - u_{yy} + \text{l.o.t.} = f, \quad \text{(1.3.8)}
\]

where l.o.t. mean lower order terms, and we call such equations elliptic and hyperbolic respectively.
Chapter 1. Introduction

What to do if one of the 2-nd derivatives is missing? We get parabolic equations

\[ u_{xx} - cu_y + \text{l.o.t.} = f. \quad (1.3.9) \]

with \( c \neq 0 \) (we do not consider \( cu_y \) as a lower order term here) and IVP \( u|_{y=0} = g \) is well-posed in the direction of \( y > 0 \) if \( c > 0 \) and in direction \( y < 0 \) if \( c < 0 \). We can dismiss \( c = 0 \) as not-interesting.

However this classification leaves out very important Schrödinger equation

\[ u_{xx} + icu_y = 0 \quad (1.3.10) \]

with real \( c \neq 0 \). For it IVP \( u|_{y=0} = g \) is well-posed in both directions \( y > 0 \) and \( y < 0 \) but it lacks many properties of parabolic equations (like maximum principle or mollification; still it has interesting properties on its own).

3D Again, if we consider only 2-nd order equations with constant real coefficients, then in appropriate coordinates they will look like either

\[ u_{xx} + u_{yy} + u_{zz} + \text{l.o.t.} = f \quad (1.3.11) \]

or

\[ u_{xx} + u_{yy} - u_{zz} + \text{l.o.t.} = f, \quad (1.3.12) \]

and we call such equations elliptic and hyperbolic respectively.

Also we get parabolic equations like

\[ u_{xx} + u_{yy} - cu_z + \text{l.o.t.} = f. \quad (1.3.13) \]

What about

\[ u_{xx} - u_{yy} - cu_z + \text{l.o.t.} = f? \quad (1.3.14) \]

Algebraist-formalist would call it parabolic-hyperbolic but since this equation exhibits no interesting analytic properties (unless one considers lack of such properties interesting; in particular, IVP is ill-posed in both directions) it would be a perversion.

Yes, there will be Schrödinger equation

\[ u_{xx} + u_{yy} + icu_z = 0 \quad (1.3.15) \]

with real \( c \neq 0 \) but \( u_{xx} - u_{yy} + icu_z = 0 \) would also have IVP \( u|_{z=0} = g \) well-posed in both directions.
Chapter 1. Introduction

4D Here we would get also elliptic
\[ u_{xx} + u_{yy} + u_{zz} + u_{tt} + \text{l.o.t.} = f, \]  
(1.3.16)

hyperbolic
\[ u_{xx} + u_{yy} + u_{zz} - u_{tt} + \text{l.o.t.} = f, \]  
(1.3.17)
but also ultrahyperbolic
\[ u_{xx} + u_{yy} - u_{zz} - u_{tt} + \text{l.o.t.} = f, \]  
(1.3.18)
which exhibits some interesting analytic properties but these equations are way less important than elliptic, hyperbolic or parabolic.

Parabolic and Schrödinger will be here as well.

Remark 1.3.1. (a) The notions of elliptic, hyperbolic or parabolic equations are generalized to higher dimensions (trivially) and to higher-order equations, but most of the randomly written equations do not belong to any of these types and there is no reason to classify them.

(b) There is no complete classifications of PDEs and cannot be because any reasonable classification should not be based on how equation looks like but on the reasonable analytic properties it exhibits (which IVP or BVP are well-posed etc).

Equations of the variable type To make things even more complicated there are equations changing types from point to point, f.e. Tricomi equation
\[ u_{xx} + xu_{yy} = 0 \]  
(1.3.19)
which is elliptic as \( x > 0 \) and hyperbolic as \( x < 0 \) and at \( x = 0 \) has a “parabolic degeneration”. It is a toy-model describing stationary transsonic flow of gas. These equations are called equations of the variable type (a.k.a. mixed type equations).

Our purpose was not to give exact definitions but to explain a situation.

1.3.3 Scope of this Textbook

- We mostly consider linear PDE problems.
- We mostly consider well-posed problems.
- We mostly consider problems with constant coefficients.
- We do not consider numerical methods.
1.4 Origin of some equations

1.4.1 Wave equation

*Example 1.4.1.* Consider a string as a curve \( y = u(x, t) \) (so it’s shape depends on time \( t \)) with a tension \( T \) and with a linear density \( \rho \). We assume that \( |u_x| \ll 1 \).

Observe that at point \( x \) the part of the string to the left from \( x \) pulls it up with a force \( -F(x) := -Tu_x \).

![Diagram of a string with forces](image)

Indeed, the force \( T \) is directed along the curve and the slope of angle \( \theta \) between the tangent to the curve and the horizontal line is \( u_x \); so \( \sin(\theta) = u_x / \sqrt{1 + u_x^2} \) which under our assumption we can replace by \( u_x \).

*Example 1.4.1 (continued).* On the other hand, at point \( x \) the part of the string to the right from \( x \) pulls it up with a force \( F(x) := Tu_x \). Therefore the total \( y \)-component of the force applied to the segment of the string between \( J = [x_1, x_2] \) equals

\[
F(x_2) - F(x_1) = \int_J \partial_x F(x) \, dx = \int_J Tu_{xx} \, dx.
\]

According to Newton’s law it must be equal to \( \int_J \rho u_{tt} \, dx \) where \( \rho \) is the mass and \( u_{tt} \) is the acceleration of the infinitesimal segment \( [x, x + dx] \):

\[
\int_J \rho u_{tt} \, dx = \int_J Tu_{xx} \, dx.
\]

Since this equality holds for any segment \( J \), the integrands coincide:

\[
\rho u_{tt} = Tu_{xx}. \tag{1.4.1}
\]

*Example 1.4.2.* Consider a membrane as a surface \( z = u(x, y, t) \) with a tension \( T \) and with a surface density \( \rho \). We assume that \( |u_x|, |u_{yx}| \ll 1 \).
Consider a domain $D$ on the plane, its boundary $L$ and a small segment of the length $ds$ of this boundary. Then the outer domain pulls this segment up with the force $-Tn \cdot \nabla u ds$ where $n$ is the inner unit normal to this segment.

Indeed, the total force is $Td ds$ but it pulls along the surface and the slope of the surface in the direction of $n$ is approximately $\pi n \cdot ru ds$.

Therefore the total $z$-component of force applied to $D$ due to Gauss formula in dimension 2 (A.1.1) equals

$$-\int_L Tn \cdot \nabla u ds = \int_D \nabla \cdot (T \nabla u) dxdy.$$ 

According to Newton’s law it must be equal to $\int_D \rho u_{tt} dxdy$ where $\rho dxdy$ is the mass and $u_{tt}$ is the acceleration of the element of the area:

$$\int_D \rho u_{tt} dxdy = \int_D T \Delta u dx$$

because $\nabla \cdot (T \nabla u) = T \nabla \cdot \nabla u T \Delta u$. Since this equality holds for any domain, the integrands coincide:

$$\rho u_{tt} = T \Delta u. \quad (1.4.2)$$

**Example 1.4.3.** Consider a gas and let $v$ be its velocity and $\rho$ its density. Then

$$\rho v_t + \rho (v \cdot \nabla) v = -\nabla p, \quad (1.4.3)$$

$$\rho_t + \nabla \cdot (\rho v) = 0 \quad (1.4.4)$$

where $p$ is the pressure. Indeed, in (1.4.3) the left-hand expression is $\rho \frac{dv}{dt}$ (the mass per unit of the volume multiplied by acceleration) and the right hand expression is the force of the pressure; no other forces are considered.

Further, (1.4.4) is *continuity equation* which means the mass conservation since the flow of the mass through the surface element $dS$ in the direction of the normal $n$ for time $dt$ equals $\rho n \cdot v$.

**Remark 1.4.1.** According to the chain rule

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + (\nabla u) \cdot \frac{dx}{dt} = u_t + (\nabla u) \cdot v$$

is a derivative of $u$ along trajectory which does not coincide with the partial derivative $u_t$; $v \cdot \nabla u$ is called *convection term*. However in the linearization with $|v| \ll 1$ it is negligible.
Remark 1.4.2. Consider any domain $V$ with a border $\Sigma$. The flow of the gas inwards for time $dt$ equals
\[
\int_\Sigma \rho \mathbf{v} \cdot \mathbf{n} \, dSdt = - \iiint_V \nabla \cdot (\rho \mathbf{v}) \, dxdydz \times dt
\]
again due to Gauss formula (A.1.2). This equals to the increment of the mass in $V$
\[
\partial_t \iiint_V \rho \, dxdydz \times dt = \iiint_V \rho_{tt} \, dxdydz \times dt.
\]

Remark 1.4.3 (continued). Therefore
\[
- \iiint_V \nabla \cdot (\rho \mathbf{v}) \, dxdydz = \iiint_V \rho_{tt} \, dxdydz
\]
Since this equality holds for any domain $V$ we can drop integral and arrive to
\[
\rho_t + \nabla \cdot (\rho \mathbf{v}) = 0. \tag{4}
\]
Example 1.4.3 (continued). We need to add $p = p(\rho, T)$ where $T$ is the temperature, but we assume $T$ is constant. Assuming that $\mathbf{v}$, $\rho - \rho_0$ and their first derivatives are small ($\rho_0 = \text{const}$) we arrive instead to
\[
\rho_0 \mathbf{v}_t = -p'(\rho_0) \nabla \rho, \tag{1.4.5}
\]
\[
\rho_t + \rho_0 \nabla \cdot \mathbf{v} = 0 \tag{1.4.6}
\]
and then applying $\nabla \cdot$ to (1.4.5) and $\partial_t$ to (1.4.6) we arrive to
\[
\rho_{tt} = c^2 \Delta \rho \tag{1.4.7}
\]
with $c = \sqrt{p'(\rho_0)}$ is the speed of sound.
1.4.2 Diffusion equation

Example 1.4.4 (Diffusion Equation). Let $u$ be a concentration of perfume in the still air. Consider some volume $V$, then the quantity of the perfume in $V$ at time $t$ equals $\int \int \int_V u \, dx \, dy \, dz$ and its increment for time $dt$ equals

$$\int \int \int_V u_t \, dx \, dy \, dz \times dt.$$ 

On the other hand, the law of diffusion states that the flow of perfume through the surface element $dS$ in the direction of the normal $n$ for time $dt$ equals $-k \nabla u \cdot n \, dS \, dt$ where $k$ is a diffusion coefficient and therefore the flow of the perfume into $V$ from outside for time $dt$ equals

$$- \int \int \int_{\Sigma} k \nabla u \cdot n \, dS \times dt = \int \int \int_V \nabla \cdot (k \nabla u) \, dx \, dy \, dz \times dt$$

due to Gauss formula (A.1.1.2).

Therefore if there are neither sources nor sinks (negative sources) in $V$ these two expression must be equal

$$\int \int \int_V u_t \, dx \, dy \, dz = \int \int \int_V \nabla \cdot (k \nabla u) \, dx \, dy \, dz$$

where we divided by $dt$. Since these equalities must hold for any volume the integrands must coincide and we arrive to continuity equation:

$$u_t = \nabla \cdot (k \nabla u).$$

(1.4.8)

If $k$ is constant we get

$$u_t = k \Delta u.$$  

(1.4.9)

Example 1.4.5 (Heat Equation). Consider heat propagation. Let $T$ be a temperature. Then the heat energy contained in the volume $V$ equals $\int \int \int_V Q(T) \, dx \, dy \, dz$ where $Q(T)$ is a heat energy density. On the other hand, the heat flow (the flow of the heat energy) through the surface element $dS$ in the direction of the normal $n$ for time $dt$ equals $-k \nabla T \cdot n \, dS \, dt$ where $k$ is a thermoconductivity coefficient. Applying the same arguments as above we arrive to

$$Q_t = \nabla \cdot (k \nabla T).$$

(1.4.10)
which we rewrite as

\[ cT_t = \nabla \cdot (k \nabla T). \]  

(1.4.11)

where \( c = \frac{\partial Q}{\partial T} \) is a thermocapacity coefficient.

If both \( c \) and \( k \) are constant we get

\[ cT_t = k \Delta T. \]  

(1.4.12)

In the real life \( c \) and \( k \) depend on \( T \). Further, \( Q(T) \) has jumps at phase transition temperature. For example to melt an ice to a water (both at 0°) requires a lot of heat and to boil the water to a vapour (both at 100°) also requires a lot of heat.

### 1.4.3 Laplace equation

*Example* 1.4.6. Considering all examples above and assuming that unknown function does not depend on \( t \) (and thus replacing corresponding derivatives by 0), we arrive to the corresponding stationary equations the simplest of which is Laplace equation

\[ \Delta u = 0. \]  

(1.4.13)

*Example* 1.4.7. In the theory of complex variables one studies holomorphic (analytic) complex-valued function \( f(z) \) satisfying a Cauchy-Riemann equation \( \partial_z f = 0 \). Here \( z = x + iy, f = u(x, y) + iv(x, y) \) with real-valued \( u = u(x, y) \) and \( v = v(x, y) \) and \( \partial_z = \frac{1}{2}(\partial_x + i\partial_y) \); then this equation could be rewritten as

\[ \partial_x u - \partial_y v = 0, \]  

(1.4.14)

\[ \partial_x v + \partial_y u = 0, \]  

(1.4.15)

which imply that both \( u, v \) satisfy Laplace equation (1.4.13).

Indeed, differentiating the first equation by \( x \) and the second by \( y \) and adding we get \( \Delta u = 0 \), and differentiating the second equation by \( x \) and the first one by \( y \) and subtracting we get \( \Delta v = 0 \).
Chapter 1. Introduction

Problems to Chapter 1

Problem 1. Consider first order equations and determine if they are linear homogeneous, linear inhomogeneous, or nonlinear ($u$ is an unknown function); for nonlinear equations, indicate if they are also semilinear, or quasilinear:

\[
\begin{align*}
  u_t + xu_x &= 0; \\
  u_t + xu_x - u &= 0; \\
  u_t + u_x - u^2 &= 0; \\
  u_x^2 + u_y^2 - 1 &= 0; \\
  u_x^2 + u_y^2 + u_z^2 - 1 &= 0; \\
  u_t + uu_x &= 0; \\
  u_t + uu_x + x &= 0; \\
  u_t^2 - u_x^2 - 1 &= 0; \\
  xu_x + yu_y + zu_z &= 0; \\
  u_t + uu_x + u^2 &= 0.
\end{align*}
\]

Problem 2. Consider equations and determine their order; determine if they are linear homogeneous, linear inhomogeneous or non-linear ($u$ is an unknown function); you need to provide the most precise (that means the narrow, but still correct) description:

\[
\begin{align*}
  u_t + (1 + x^2)u_{xx} &= 0; \\
  u_t + u_{xxx} &= 0; \\
  u_{tt} + u_{xxxx} &= 0; \\
  u_{tt} + u_{xxxx} + \sin(x) &= 0; \\
  u_{tt} + u_{xxxx} + \sin(x)\sin(u) &= 0.
\end{align*}
\]

Problem 3. Find the general solutions to the following equations:

\[
\begin{align*}
  u_{xy} &= 0; \\
  u_{xy} &= 2u_x; \\
  u_{xy} &= 2u_x + e^{x+y};
\end{align*}
\]

Hint. Introduce $v = u_x$ and find it first.

---

1) $F(x, y, u, u_x, u_y) = 0$ is non-linear unless

\[
F := au_x + bu_y + cu - f
\]

with $a = a(x, y)$, $b = b(x, y)$, $c = c(x, y)$ and $f = f(x, y)$, when it is linear homogeneous for $f(x, y) = 0$ and linear inhomogeneous otherwise. If

\[
F := au_x + bu_y - f
\]

with $a = a(x, y, u)$, $b = b(x, y, u)$ and $f = f(x, y, u)$ (so it is linear with respect to (highest order) derivatives, it is called quasilinear, and if in addition $a = a(x, y)$, $b = b(x, y)$, it is called semilinear. This definition obviously generalizes to higher dimensions and orders.
Chapter 1. Introduction

Problem 4. Find the general solutions to the following equations:

\[ uu_{xy} = u_x u_y; \quad uu_{xy} = 2u_x u_y; \quad u_{xy} = u_x u_y. \]

*Hint.* Divide two first equations by \( uu_x \) and observe that both the right and left-hand expressions are derivative with respect to \( y \) of \( \ln(u_x) \) and \( \ln(u) \) respectively. Divide the last equation by \( u_x \).

Problem 5. Find the general solutions to the following *linear homogeneous* equations:

\[ u_{xxy} = 0, \quad u_{xxy} = 0, \quad u_{xxyz} = 0, \quad u_{xx} = \sin(x) \sin(y), \quad u_{xx} = \sin(x) \sin(y), \quad u_{xx} = \sin(x) \sin(y), \quad u_{xx} = \sin(x) \sin(y), \quad u_{xy} = \sin(x) \sin(y) \sin(z), \quad u_{xyz} = \sin(x) + \sin(y) + \sin(z). \]

Problem 6. Find the general solutions to the following *overdetermined systems*:

\[
\begin{aligned}
\begin{cases}
  u_{xx} = 0, \\
  u_y = 0;
\end{cases}
\quad
\begin{cases}
  u_x = 0, \\
  u_{xz} = 0;
\end{cases}
\]

\[
\begin{aligned}
\begin{cases}
  u_{xy} = 0, \\
  u_{xz} = 0, \\
  u_{yz} = 0;
\end{cases}
\quad
\begin{cases}
  u_{xx} = 6xy, \\
  u_y = x^3;
\end{cases}
\quad
\end{aligned}
\]

\[
\begin{aligned}
\begin{cases}
  u_{xx} = 6xy, \\
  u_y = x^3.
\end{cases}
\end{aligned}
\]

*Hint.* Solve one of the equations and plugging the result to another, specify an arbitrary function (or functions) in it, and write down the final answer. Often overdetermined systems do not have solutions; i.e.

\[
\begin{aligned}
\begin{cases}
  u_x = M(x, y), \\
  u_y = N(x, y)
\end{cases}
\end{aligned}
\]

has a solution iff \( M_y - N_x = 0 \).