Chapter 1

Introduction

1.1 PDE motivations and context

The aim of this is to introduce and motivate partial differential equations (PDE). The section also places the scope of studies in APM346 within the vast universe of mathematics. A *partial differential equation (PDE)* is an *gather* involving *partial derivatives*. This is not so informative so let's break it down a bit.

1.1.1 What is a differential equation?

An *ordinary differential equation (ODE)* is an equation for a function which depends on one independent variable which involves the independent variable, the function, and derivatives of the function:

$$F(t, u(t), u'(t), u^{(2)}(t), u^{(3)}(t), \dots, u^{(m)}(t)) = 0.$$

This is an example of an ODE of *order* m where m is a highest order of the derivative in the equation. Solving an equation like this on an interval $t \in [0, T]$ would mean finding a function $t \mapsto u(t) \in \mathbb{R}$ with the property that u and its derivatives satisfy this equation for all values $t \in [0, T]$.

The problem can be enlarged by replacing the real-valued u by a vector-valued one $\boldsymbol{u}(t) = (u_1(t), u_2(t), \dots, u_N(t))$. In this case we usually talk about *system of ODEs*.

Even in this situation, the challenge is to find functions depending upon exactly one variable which, together with their derivatives, satisfy the equation.

What is a partial derivative?

When you have function that depends upon several variables, you can differentiate with respect to either variable while holding the other variable constant. This spawns the idea of *partial derivatives.* As an example, consider a function depending upon two real variables taking values in the reals:

$$u\colon \mathbb{R}^n \to \mathbb{R}.$$

As $n = 2$ we sometimes visualize a function like this by considering its *graph* viewed as a surface in $\mathbb{R}^3$ given by the collection of points

$$\{(x, y, z) \in \mathbb{R}^3 : z = u(x, y)\}.$$

We can calculate the derivative with respect to x while holding y fixed. This leads to u_x, also expressed as $\partial_x u$, $\frac{\partial u}{\partial x}$, and $\frac{\partial}{\partial x}u$. Similarly, we can hold x fixed and differentiate with respect to y.

What is PDE?

A *partial differential equation* is an equation for a function which depends on more than one independent variable which involves the independent variables, the function, and partial derivatives of the function:

$$F(x, y, u(x, y), u_x(x, y), u_y(x, y), u_{xx}(x, y), u_{xy}(x, y), u_{yx}(x, y), u_{yy}(x, y)) = 0.$$

This is an example of a *PDE of order 2.* Solving an equation like this would mean finding a function $(x, y) \to u(x, y)$ with the property that u and its derivatives satisfy this equation for all admissible arguments.

Similarly to ODE case this problem can be enlarged by replacing the real-valued u by a vector-valued one $\boldsymbol{u}(t) = (u_1(x, y), u_2(x, y), \dots, u_N(x, y)$. In this case we usually talk about *system of PDEs.*

Where PDEs are coming from?

PDEs are often referred as *Equations of Mathematical Physics* (or *Mathematical Physics* but it is incorrect as Mathematical Physics is now a separate field of mathematics) because many of PDEs are coming from different domains of physics (acoustics, optics, elasticity, hydro and aerodynamics, electromagnetism, quantum mechanics, seismology etc).

However PDEs appear in other fields of science as well (like quantum chemistry, chemical kinetics); some PDEs are coming from economics and financial mathematics, or computer science.

Many PDEs are originated in other fields of mathematics.

Examples of PDEs

(Some are actually systems)

Simplest first order equation

$$u_x = 0.$$

Transport equation

$$u_t + cu_x = 0.$$

$\bar\partial$ equation

$$\bar\partial f := \frac{1}{2}(f_x + if_y) = 0,$$

($\bar\partial$ is known as "di-bar" or Wirtinger derivatives), or as $f = u + iv$

$$\begin{cases} u_x - v_y = 0, \\ u_y + v_x = 0. \end{cases}$$

Those who took *Complex variables* know that those are *Cauchy-Riemann equations.*

Laplace's equation (in 2D)

$$\Delta u := u_{xx} + u_{yy} = 0$$

or similarly in the higher dimensions.

Heat equation

$$u_t = k\Delta u;$$

(The expression Δ is called the *Laplacian* (*Laplace operator*) and is defined as $\partial_x^2 + \partial_y^2 + \partial_z^2$ on $\mathbb{R}^3$).

Schrödinger equation (quantum mechanics)

$$i\hbar\partial_t\psi = \bigl(-\frac{\hbar^2}{2m}\Delta + V\bigr)\psi.$$

Here ψ is a complex-valued function.

Wave equation

$$u_{tt} - c^2\Delta u = 0;$$

sometimes $\Box := c^{-2}\partial_t^2 - \Delta$ is called (*d'Alembertian* or *d'Alembert operator*). It appears in elasticity, acoustics, electromagnetism and so on.

One-dimensional wave equation

$$u_{tt} - c^2 u_{xx} = 0$$

often is called *string equation* and describes f.e. a vibrating string.

Oscillating rod or plate (elasticity) Equation of vibrating rod (with one spatial variable)

$$u_{tt} + Ku_{xxxx} = 0$$

or vibrating plate (with two spatial variables)

$$u_{tt} + K\Delta^2 u = 0.$$

Maxwell equations (electromagnetism) in vacuum

$$\begin{cases} \boldsymbol{E}_t - c\nabla \times \boldsymbol{H} = 0, \\ \boldsymbol{H}_t + c\nabla \times \boldsymbol{E} = 0, \\ \nabla \cdot \boldsymbol{E} = \nabla \cdot \boldsymbol{H} = 0. \end{cases}$$

Here $\boldsymbol{E}$ and $\boldsymbol{H}$ are vectors of electric and magnetic intensities, so the first two lines are actually 6×6 system. The third line means two more equations, and we have 8×6 system. Such systems are called *overdetermined*.

Dirac equations (relativistic quantum mechanics)

$$i\hbar\partial_t\psi = \bigl(\beta mc^2 - \sum_{1\le k\le 3} ic\hbar\alpha_k\partial_{x_k}\bigr)\psi$$

with *Dirac* 4×4-*matrices* $\alpha_1, \alpha_2, \alpha_3, \beta$. Here ψ is a complex 4-vector, so in fact we have 4×4 system.

Elasticity equations (homogeneous and isotropic)

$$\boldsymbol{u}_{tt} = \lambda\Delta\boldsymbol{u} + \mu\nabla(\nabla\cdot\boldsymbol{u}).$$

homogeneous means "the same in all places" (an opposite is called *inhomogeneous*) and *isotropic* means "the same in all directions" (an opposite is called *anisotropic*).

Navier-Stokes equation (hydrodynamics for incompressible liquid)

$$\begin{cases} \rho\boldsymbol{v}_t + (\boldsymbol{v}\cdot\nabla)\rho\boldsymbol{v} - \nu\Delta\boldsymbol{v} = -\nabla p,\\ \nabla\cdot\boldsymbol{v} = 0, \end{cases}$$

where ρ is a (constant) density, $\boldsymbol{v}$ is a velocity and p is the pressure; when viscosity $\nu = 0$ we get *Euler equation*

$$\begin{cases} \rho\boldsymbol{v}_t + (\boldsymbol{v}\cdot\nabla)\rho\boldsymbol{v} = -\nabla p,\\ \nabla\cdot\boldsymbol{v} = 0. \end{cases}$$

Both of them are 4×4 systems.

Yang-Mills equation (elementary particles theory)

$$\begin{aligned} &\partial_{x_j}F_{jk} + [A_j, F_{jk}] = 0,\\ &F_{jk} \coloneqq \partial_{x_j}A_k - \partial_{x_k}A_j + [A_j, A_k], \end{aligned}$$

where A_k are traceless skew-Hermitian matrices. Their matrix elements are unknown functions. This is a 2nd order system.

Einstein equation for general relativity

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu},$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}$ is the *Einstein tensor*, $g_{\mu\nu}$ is the *metric tensor* (unknown functions), $R_{\mu\nu}$ is the *Ricci curvature tensor*, and R is the *scalar curvature*, $T_{\mu\nu}$ is the *stress–energy tensor*, Λ is the *cosmological constant* and κ is the *Einstein gravitational constant*. Components of Ricci curvature tensor are expressed through the components of the metric tensor, their first and second derivatives. This is a 2nd order system.

Black-Scholes equation Black-Scholes Equation (Financial mathematics) is a partial differential equation (PDE) governing the price evolution of a European call or European put under the Black–Scholes model. Broadly speaking, the term may refer to a similar PDE that can be derived for a variety of options, or more generally, derivatives:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

where V is the price of the option as a function of stock price S and time t, r is the risk-free interest rate, and σ is the volatility of the stock.

Do not ask me what this means!

Remark 1.1.1. (a) Some of these examples are actually not single PDEs but the systems of PDEs.

(b) In all this examples there are *spatial variables* x, y, z and often *time variable* t but it is not necessarily so in all PDEs. Equations, not including time, are called *stationary* (an opposite is called *nonstationary*).

(c) Equations could be of different order with respect to different variables and it is important. However if not specified the order of equation is the highest order of the derivatives invoked.

(d) In this class we will deal mainly with the wave equation, heat equation and Laplace equation in their simplest forms.

1.2 Initial and boundary value problems

1.2.1 Problems for PDEs

We know that solutions of ODEs typically depend on one or several constants. For PDEs situation is more complicated. Consider simplest equations

$$u_x = 0, \tag{1.2.1}$$
$$v_{xx} = 0 \tag{1.2.2}$$
$$w_{xy} = 0 \tag{1.2.3}$$

with $u = u(x, y)$, $v = v(x, y)$ and $w = w(x, y)$. Equation (1.2.1) could be treaded as an ODE with respect to x and its solution is a constant but this is not a genuine constant as *it is constant only with respect to x and can depend on other variables*; so $u(x, y) = \phi(y)$.

Meanwhile, for solution of (1.2.2) we have $v_x = \phi(y)$ where ϕ is an arbitrary function of one variable and it could be considered as ODE with respect to x again; then $v(x, y) = \phi(y)x + \psi(y)$ where $\psi(y)$ is another arbitrary function of one variable.

Finally, for solution of (1.2.3) we have $w_y = \phi(y)$ where ϕ is an arbitrary function of one variable and it could be considered as ODE with respect to y; then $(w - g(y))_y = 0$ where $g(y) = \int \phi(y)\, dy$, and therefore $w - g(y) = f(x) \implies w(x, y) = f(x) + g(y)$ where f, g are *arbitrary functions* of one variable.

Considering these equations again but assuming that $u = u(x, y, z)$, $v = v(x, y, z)$ we arrive to $u = \phi(y, z)$, $v = \phi(y, z)x + \psi(y, z)$ and $w = f(x, z) + g(y, z)$ where f, g are arbitrary functions of two variables.

Solutions to PDEs typically depend not on several arbitrary constants but on one or several arbitrary functions of $n - 1$ variables. For more complicated equations this dependence could be much more complicated and implicit. To select a right solutions we need to use some extra conditions.

The sets of such conditions are called *Problems*. Typical problems are

(a) IVP (*Initial Value Problem*): one of variables is interpreted as *time* t and conditions are imposed at some moment; f.e. $u|_{t=t_0} = u_0$;

(b) BVP (*Boundary Value Problem*) conditions are imposed on the boundary of the spatial domain Ω: f.e. $u|_{\partial\Omega} = \phi$ where $\partial\Omega$ is a boundary of Ω and ϕ is defined on $\partial\Omega$;

(c) IVBP (*Initial-Boundary Value Problem* a.k.a. *mixed problem*): one of variables is interpreted as *time* t and some conditions are imposed at some moment but other conditions are imposed on the boundary of the spatial domain.

Remark 1.2.1. In the course of ODEs students usually consider IVP only. F.e. for the second-order equation like

$$u_{xx} + a_1 u_x + a_2 u = f(x)$$

such problem is $u|_{x=x_0} = u_0$, $u_x|_{x=x_0} = u_1$. However one could consider BVPs like

$$(\alpha_1 u_x + \beta_1 u)|_{x=x_1} = \phi_1,$$
$$(\alpha_2 u_x + \beta_2 u)|_{x=x_2} = \phi_2,$$

where solutions are sought on the interval $[x_1, x_2]$. Such are covered in advanced chapters of some of ODE textbooks (but not covered by a typical ODE class). We will need to cover such problems later in this class.

1.2.2 Notion of "well-posedness"

We want that our PDE (or the system of PDEs) together with all these conditions satisfied the following requirements:

(a) Solutions must exist for all right-hand expressions (in equations and conditions)–*existence*;

(b) Solution must be unique–*uniqueness*;

(c) Solution must depend on these right-hand expressions continuously, which means that small changes in the right-hand expressions lead to small changes in the solution.

Such problems are called *well-posed*. PDEs are usually studied together with the problems which are well-posed for these PDEs. Different types of PDEs *admit* different problems.

Sometimes however one needs to consider *ill-posed* problems. In particular, *inverse problems* of PDEs are almost always ill-posed.

1.3 Classification of equations

1.3.1 Linear and nonlinear equations

Equations of the form

$$Lu = f(\boldsymbol{x}) \tag{1.3.1}$$

where Lu is a partial differential expression linear with respect to unknown function u is called *linear equation* (or *linear system*). This equation is *linear homogeneous equation* if $f = 0$ and *linear inhomogeneous equation* otherwise. For example,

$$Lu := a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + a_1u_x + a_2u_y + au = f(\boldsymbol{x}) \tag{1.3.2}$$

is linear; if all coefficients a_{jk}, a_j, a are constant, we call it *linear equation with constant coefficients*; otherwise we talk about *variable coefficients*.

Otherwise equation is called *nonlinear*. However there is a more subtle classification of such equations. Equations of the type (1.3.1), where the right-hand expression f depends on the solution and its lower-order derivatives, are called *semilinear*, equations where both coefficients and right-hand expression depend on the solution and its lower-order derivatives are called *quasilinear*. For example,

$$Lu := a_{11}(x,y)u_{xx} + 2a_{12}(x,y)u_{xy} + a_{22}(x,y)u_{yy} = f(x,y,u,u_x,u_y) \tag{1.3.3}$$

is semilinear, and

$$Lu := a_{11}(x,y,u,u_x,u_y)u_{xx} + 2a_{12}(x,y,u,u_x,u_y)u_{xy}+ \\ a_{22}(x,y,u,u_x,u_y)u_{yy} = f(x,y,u,u_x,u_y) \tag{1.3.4}$$

is quasilinear, while

$$F(x,y,u,u_x,u_y,u_{xx},u_{xy},u_{yx}) = 0 \tag{1.3.5}$$

is general nonlinear.

1.3.2 Elliptic, hyperbolic and parabolic equations

General Consider second order equation (1.3.2):

$$Lu := \sum_{1\le i,j\le n} a_{ij}u_{x_ix_j} + \text{l.o.t.} = f(\boldsymbol{x}) \tag{1.3.6}$$

where l.o.t. means *lower order terms* (that is, terms with u and its lower order derivatives) with $a_{ij} = a_{ji}$. Let us change variables $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{x}')$. Then the *matrix of principal coefficients*

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$$

in the new coordinate system becomes $A' = Q^* A Q$ where $Q = T^{*\,-1}$ and $T = \left(\frac{\partial x_i}{\partial x'_j}\right)_{i,j=1,\dots,n}$ is a Jacobi matrix. The proof easily follows from the *chain rule* (Calculus II).

Therefore if the principal coefficients are real and constant, by a linear change of variables matrix of the principal coefficients could be reduced to the diagonal form, where diagonal elements could be either 1, or -1 or 0. Multiplying equation by -1 if needed we can assume that there are at least as many 1 as -1. In particular, for $n = 2$ the principal part becomes either $u_{xx} + u_{yy}$, or $u_{xx} - u_{yy}$, or u_{xx} and such equations are called *elliptic*, *hyperbolic*, and *parabolic* respectively (there will be always second derivative since otherwise it would be the first order equation).

This terminology comes from the curves of the second order *conical sections*: if $a_{11}a_{22} - a_{12}^2 > 0$ equation

$$a_{11}\xi^2 + 2a_{12}\xi\eta + a_{22}\eta^2 + a_1\xi + a_2\eta = c$$

generically defines an ellipse, if $a_{11}a_{22} - a_{12}^2 < 0$ this equation generically defines a hyperbole and if $a_{11}a_{22} - a_{12}^2 = 0$ it defines a parabole.

Let us consider equations in different dimensions:

2D If we consider only 2-nd order equations with constant real coefficients then in appropriate coordinates they will look like either

$$u_{xx} + u_{yy} + \text{l.o.t} = f \tag{1.3.7}$$

or

$$u_{xx} - u_{yy} + \text{l.o.t.} = f, \tag{1.3.8}$$

where l.o.t. mean *lower order terms*, and we call such equations *elliptic* and *hyperbolic* respectively.

What to do if one of the 2-nd derivatives is missing? We get *parabolic equations*

$$u_{xx} - cu_y + \text{l.o.t.} = f. \tag{1.3.9}$$

with $c \neq 0$ (we do not consider cu_y as a lower order term here) and IVP $u|_{y=0} = g$ is well-posed in the direction of $y > 0$ if $c > 0$ and in direction $y < 0$ if $c < 0$. We can dismiss $c = 0$ as not-interesting.

However this classification leaves out very important Schrödinger equation

$$u_{xx} + icu_y = 0 \tag{1.3.10}$$

with real $c \neq 0$. For it IVP $u|_{y=0} = g$ is well-posed in both directions $y\,0$ and $y < 0$ but it lacks many properties of parabolic equations (like maximum principle or mollification; still it has interesting properties on its own).

3D Again, if we consider only 2-nd order equations with constant real coefficients, then in appropriate coordinates they will look like either

$$u_{xx} + u_{yy} + u_{zz} + \text{l.o.t} = f \tag{1.3.11}$$

or

$$u_{xx} + u_{yy} - u_{zz} + \text{l.o.t.} = f, \tag{1.3.12}$$

and we call such equations *elliptic* and *hyperbolic* respectively.

Also we get *parabolic equations* like

$$u_{xx} + u_{yy} - cu_z + \text{l.o.t.} = f. \tag{1.3.13}$$

What about

$$u_{xx} - u_{yy} - cu_z + \text{l.o.t.} = f? \tag{1.3.14}$$

Algebraist-formalist would call it parabolic-hyperbolic but since this equation exhibits no interesting analytic properties (unless one considers lack of such properties interesting; in particular, IVP is ill-posed in both directions) it would be a perversion.

Yes, there will be Schrödinger equation

$$u_{xx} + u_{yy} + icu_z = 0 \tag{1.3.15}$$

with real $c \neq 0$ but $u_{xx} - u_{yy} + icu_z = 0$ would also have IVP $u|_{z=0} = g$ well-posed in both directions.

4D Here we would get also *elliptic*

$$u_{xx} + u_{yy} + u_{zz} + u_{tt} + \text{l.o.t.} = f, \tag{1.3.16}$$

hyperbolic

$$u_{xx} + u_{yy} + u_{zz} - u_{tt} + \text{l.o.t.} = f, \tag{1.3.17}$$

but also *ultrahyperbolic*

$$u_{xx} + u_{yy} - u_{zz} - u_{tt} + \text{l.o.t.} = f, \tag{1.3.18}$$

which exhibits some interesting analytic properties but these equations are way less important than elliptic, hyperbolic or parabolic.

Parabolic and Schrödinger will be here as well.

Remark 1.3.1. (a) The notions of elliptic, hyperbolic or parabolic equations are generalized to higher dimensions (trivially) and to higher-order equations, but most of the randomly written equations do not belong to any of these types and there is no reason to classify them.

(b) There is no complete classifications of PDEs and cannot be because any reasonable classification should not be based on how equation looks like but on the reasonable analytic properties it exhibits (which IVP or BVP are well-posed etc).

Equations of the variable type To make things even more complicated there are equations changing types from point to point, f.e. Tricomi equation

$$u_{xx} + xu_{yy} = 0 \tag{1.3.19}$$

which is elliptic as $x > 0$ and hyperbolic as $x < 0$ and at $x = 0$ has a "parabolic degeneration". It is a toy-model describing stationary transsonic flow of gas. These equations are called *equations of the variable type* (a.k.a. *mixed type equations*).

Our purpose was not to give exact definitions but to explain a situation.

1.3.3 Scope of this Textbook

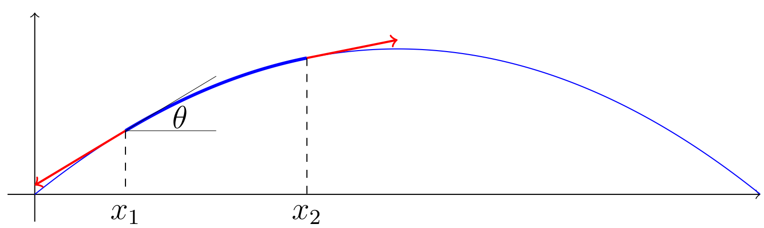
- We mostly consider *linear* PDE problems.
- We mostly consider *well-posed problems.*
- We mostly consider problems with *constant coefficients.*
- We do not consider *numerical methods.*

1.4 Origin of some equations

1.4.1 Wave equation

Example 1.4.1. Consider a string as a curve $y = u(x,t)$ (so it's shape depends on time t) with a tension T and with a linear density ρ. We assume that $|u_x| \ll 1$.

Observe that at point x the part of the string to the left from x pulls it up with a force $-F(x) := -Tu_x$.



Indeed, the force T is directed along the curve and the slope of angle θ between the tangent to the curve and the horizontal line is u_x; so $\sin(\theta) = u_x/\sqrt{1+u_x^2}$ which under our assumption we can replace by u_x.

Example 1.4.1 (continued). On the other hand, at point x the part of the string to the right from x pulls it up with a force $F(x) := Tu_x$. Therefore the total y-component of the force applied to the segment of the string between $J = [x_1, x_2]$ equals

$$F(x_2) - F(x_1) = \int_J \partial_x F(x)\,dx = \int_J Tu_{xx}\,dx.$$

According to Newton's law it must be equal to $\int_J \rho u_{tt}\,dx$ where ρdx is the mass and u_{tt} is the acceleration of the infinitesimal segment $[x, x+dx]$:

$$\int_J \rho u_{tt}\,dx = \int_J Tu_{xx}\,dx.$$

Since this equality holds for any segment J, the integrands coincide:

$$\rho u_{tt} = Tu_{xx}. \tag{1.4.1}$$

Example 1.4.2. Consider a membrane as a surface $z = u(x,y,t)$ with a tension T and with a surface density ρ. We assume that $|u_x|, |u_{yy}| \ll 1$.

Consider a domain D on the plane, its boundary L and a small segment of the length ds of this boundary. Then the outer domain pulls this segment up with the force $-T\boldsymbol{n}\cdot\nabla u\,ds$ where $\boldsymbol{n}$ is the *inner* unit normal to this segment.

Indeed, the total force is $T\,ds$ but it pulls along the surface and the slope of the surface in the direction of $\boldsymbol{n}$ is $\approx \boldsymbol{n}\cdot\nabla u$.

Therefore the total z-component of force applied to D due to Gauss formula in dimension 2 (A.1.1) equals

$$-\int_L T\boldsymbol{n}\cdot\nabla u\,ds = \iint_D \nabla\cdot(T\nabla u)\,dxdy.$$

According to Newton's law it must be equal to $\iint_D \rho u_{tt}\,dxdy$ where $\rho dxdy$ is the mass and u_{tt} is the acceleration of the element of the area:

$$\iint_D \rho u_{tt}\,dxdy = \iint_D T\Delta u\,dx$$

because $\nabla\cdot(T\nabla u) = T\nabla\cdot\nabla u T\Delta u$. Since this equality holds for any domain, the integrands coincide:

$$\rho u_{tt} = T\Delta u. \tag{1.4.2}$$

Example 1.4.3. Consider a gas and let $\boldsymbol{v}$ be its velocity and ρ its density. Then

$$\rho\boldsymbol{v}_t + \rho(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\nabla p, \tag{1.4.3}$$

$$\rho_t + \nabla\cdot(\rho\boldsymbol{v}) = 0 \tag{1.4.4}$$

where p is the pressure. Indeed, in (1.4.3) the left-hand expression is $\rho\dfrac{d\boldsymbol{v}}{dt}$ (the mass per unit of the volume multiplied by acceleration) and the right hand expression is the force of the pressure; no other forces are considered.

problem Further, (1.4.4) is *continuity equation* which means the mass conservation since the flow of the mass through the surface element dS in the direction of the normal $\boldsymbol{n}$ for time dt equals $\rho\boldsymbol{n}\cdot\boldsymbol{v}$.

Remark 1.4.1. According to the chain rule

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + (\nabla u)\cdot\frac{d\boldsymbol{x}}{dt} = u_t + (\nabla u)\cdot\boldsymbol{v}$$

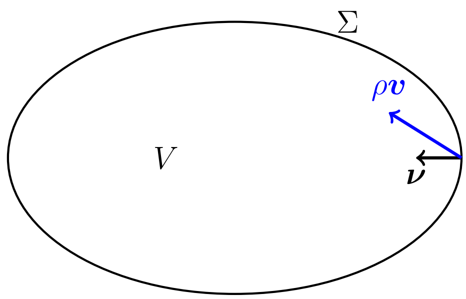
is a *derivative of* u *along trajectory* which does not coincide with the *partial derivative* u_t; $\boldsymbol{v}\cdot\nabla u$ is called *convection term*. However in the linearization with $|\boldsymbol{v}|\ll 1$ it is negligible.

Remark 1.4.2. Consider any domain V with a border Σ. The flow of the gas inwards for time dt equals

$$\iint_{\Sigma} \rho \boldsymbol{v}\cdot\boldsymbol{n}\, dSdt = -\iiint_V \nabla\cdot(\rho\boldsymbol{v})\, dxdydz\times dt$$

again due to Gauss formula (A.1.2). This equals to the increment of the mass in V

$$\partial_t \iiint_V \rho\, dxdydz\times dt = \iiint_V \rho_{tt}\, dxdydz\times dt.$$



Remark 1.4.3 (continued). Therefore

$$-\iiint_V \nabla\cdot(\rho\boldsymbol{v})\, dxdydz = \iiint_V \rho_{tt}\, dxdydz$$

Since this equality holds for *any* domain V we can drop integral and arrive to

$$\rho_t + \nabla\cdot(\rho\boldsymbol{v}) = 0. \tag{4}$$

Example 1.4.3 (continued). We need to add $p = p(\rho, T)$ where T is the temperature, but we assume T is constant. Assuming that $\boldsymbol{v}$, $\rho-\rho_0$ and their first derivatives are small ($\rho_0 = \text{const}$) we arrive instead to

$$\rho_0\boldsymbol{v}_t = -p'(\rho_0)\nabla\rho, \tag{1.4.5}$$
$$\rho_t + \rho_0\nabla\cdot\boldsymbol{v} = 0 \tag{1.4.6}$$

and then applying $\nabla\cdot$ to (1.4.5) and ∂_t to (1.4.6) we arrive to

$$\rho_{tt} = c^2\Delta\rho \tag{1.4.7}$$

with $c = \sqrt{p'(\rho_0)}$ is *the speed of sound*.

1.4.2 Diffusion equation

Example 1.4.4 (Diffusion Equation). Let u be a concentration of perfume in the still air. Consider some volume V, then the quantity of the perfume in V at time t equals $\iiint_V u\,dxdydz$ and its increment for time dt equals

$$\iiint_V u_t\,dxdydz \times dt.$$

On the other hand, the law of diffusion states that the flow of perfume through the surface element dS in the direction of the normal $\boldsymbol{n}$ for time dt equals $-k\nabla u\cdot\boldsymbol{n}\,dSdt$ where k is a *diffusion coefficient* and therefore the flow of the perfume into V from outside for time dt equals

$$-\iint_\Sigma k\nabla u\cdot\boldsymbol{n},dS \times dt = \iiint_V \nabla\cdot(k\nabla u)\,dxdydz \times dt$$

due to Gauss formula (A.1.1.2).

Therefore if there are neither sources nor sinks (negative sources) in V these two expression must be equal

$$\iiint_V u_t\,dxdydz = \iiint_V \nabla\cdot(k\nabla u)\,dxdydz$$

where we divided by dt. Since these equalities must hold for any volume the integrands must coincide and we arrive to *continuity equation*:

$$u_t = \nabla\cdot(k\nabla u). \tag{1.4.8}$$

If k is constant we get

$$u_t = k\Delta u. \tag{1.4.9}$$

Example 1.4.5 (Heat Equation). Consider heat propagation. Let T be a temperature. Then the heat energy contained in the volume V equals $\iiint_V Q(T)\,dxdydz$ where $Q(T)$ is a *heat energy density*. On the other hand, the heat flow (the flow of the heat energy) through the surface element dS in the direction of the normal $\boldsymbol{n}$ for time dt equals $-k\nabla T\cdot\boldsymbol{n}\,dSdt$ where k is a *thermoconductivity coefficient*. Applying the same arguments as above we arrive to

$$Q_t = \nabla\cdot(k\nabla T). \tag{1.4.10}$$

which we rewrite as

$$cT_t = \nabla \cdot (k\nabla T). \tag{1.4.11}$$

where $c = \frac{\partial Q}{\partial T}$ is a *thermocapacity coefficient*.

If both c and k are constant we get

$$cT_t = k\Delta T. \tag{1.4.12}$$

In the real life c and k depend on T. Further, $Q(T)$ has jumps at *phase transition temperature*. For example to melt an ice to a water (both at 0°) requires a lot of heat and to boil the water to a vapour (both at 100°) also requires a lot of heat.

1.4.3 Laplace equation

Example 1.4.6. Considering all examples above and assuming that unknown function does not depend on t (and thus replacing corresponding derivatives by 0), we arrive to the corresponding *stationary equations* the simplest of which is Laplace equation

$$\Delta u = 0. \tag{1.4.13}$$

Example 1.4.7. In the theory of complex variables one studies holomorphic (analytic) *complex-valued* function $f(z)$ satisfying a Cauchy-Riemann equation $\partial_{\bar{z}} f = 0$. Here $z = x + iy$, $f = u(x, y) + iv(x, y)$ with *real-valued* $u = u(x, y)$ and $v = v(x, y)$ and $\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$; then this equation could be rewritten as

$$\partial_x u - \partial_y v = 0, \tag{1.4.14}$$
$$\partial_x v + \partial_y u = 0, \tag{1.4.15}$$

which imply that both u, v satisfy Laplace equation (1.4.13).

Indeed, differentiating the first equation by x and the second by y and adding we get $\Delta u = 0$, and differentiating the second equation by x and the first one by y and subructing we get $\Delta v = 0$.

Problems to Chapter 1

Problem 1. Consider first order equations and determine if they are linear homogeneous, linear inhomogeneous, or nonlinear (u is an unknown function); for nonlinear equations, indicate if they are also semilinear, or quasilinear[1)]:

$$u_t + xu_x = 0; \qquad u_t + uu_x = 0;$$
$$u_t + xu_x - u = 0; \qquad u_t + uu_x + x = 0;$$
$$u_t + u_x - u^2 = 0; \qquad u_t^2 - u_x^2 - 1 = 0;$$
$$u_x^2 + u_y^2 - 1 = 0; \qquad xu_x + yu_y + zu_z = 0;$$
$$u_x^2 + u_y^2 + u_z^2 - 1 = 0; \qquad u_t + u_x^2 + u_y^2 = 0.$$

Problem 2. Consider equations and determine their order; determine if they are linear homogeneous, linear inhomogeneous or non-linear (u is an unknown function); you need to provide *the most precise* (that means the narrow, but still correct) description:

$$u_t + (1 + x^2)u_{xx} = 0; \qquad u_t - (1 + u^2)u_{xx} = 0;$$
$$u_t + u_{xxx} = 0, \qquad u_t + uu_x + u_{xxx} = 0;$$
$$u_{tt} + u_{xxxx} = 0; \qquad u_{tt} + u_{xxxx} + u = 0;$$
$$u_{tt} + u_{xxxx} + \sin(x) = 0; \qquad u_{tt} + u_{xxxx} + \sin(x)\sin(u) = 0.$$

Problem 3. Find the general solutions to the following equations:

$$u_{xy} = 0; \qquad u_{xy} = 2u_x;$$
$$u_{xy} = e^{x+y} \qquad u_{xy} = 2u_x + e^{x+y}.$$

Hint. Introduce $v = u_x$ and find it first.

[1)] $F(x, y, u, u_x, u_y) = 0$ is *non-linear* unless

$$F := au_x + bu_y + cu - f \tag{1..16}$$

with $a = a(x, y)$, $b = b(x, y)$, $c = c(x, y)$ and $f = f(x, y)$, when it is *linear homogeneous* for $f(x, y) = 0$ and *linear inhomogeneous* otherwise. If

$$F := au_x + bu_y - f \tag{1..17}$$

with $a = a(x, y, u)$, $b = b(x, y, u)$ and $f = f(x, y, u)$ (so it is linear with respect to (highest order) derivatives, it is called *quasilinear*, and if in addition $a = a(x, y)$, $b = b(x, y)$, it is called *semilinear*. This definition obviously generalizes to higher dimensions and orders.

Problem 4. Find the general solutions to the following equations:

$$uu_{xy} = u_x u_y; \qquad uu_{xy} = 2u_x u_y; \qquad u_{xy} = u_x u_y.$$

Hint. Divide two first equations by uu_x and observe that both the right and left-hand expressions are derivative with respect to y of $\ln(u_x)$ and $\ln(u)$ respectively. Divide the last equation by u_x.

Problem 5. Find the general solutions to the following *linear homogeneous equations*:

$$\begin{aligned}
&u_{xxy} = 0, && u_{xxyy} = 0,\\
&u_{xxxy} = 0, && u_{xyz} = 0,\\
&u_{xyzz} = 0, && u_{xxy} = \sin(x)\sin(y),\\
&u_{xxy} = \sin(x) + \sin(y), && u_{xxyy} = \sin(x)\sin(y),\\
&u_{xxyy} = \sin(x) + \sin(y), && u_{xxxy} = \sin(x)\sin(y),\\
&u_{xxxy} = \sin(x) + \sin(y), && u_{xyz} = \sin(x)\sin(y)\sin(z),\\
&u_{xyz} = \sin(x) + \sin(y) + \sin(z), && u_{xyz} = \sin(x) + \sin(y)\sin(z).
\end{aligned}$$

Problem 6. Find the general solutions to the following *overdetermined systems*:

$$\begin{cases} u_{xx} = 0,\\ u_y = 0; \end{cases} \qquad \begin{cases} u_{xy} = 0,\\ u_{xz} = 0; \end{cases}$$

$$\begin{cases} u_{xy} = 0,\\ u_{xz} = 0,\\ u_{yz} = 0; \end{cases} \qquad \begin{cases} u_{xx} = 6xy,\\ u_y = x^3; \end{cases}$$

$$\begin{cases} u_{xx} = 6xy,\\ u_y = -x^3. \end{cases}$$

Hint. Solve one of the equations and plugging the result to another, specify an arbitrary function (or functions) in it, and write down the final answer. Often overdetermined systems do not have solutions; f.e.

$$\begin{cases} u_x = M(x,y),\\ u_y = N(x,y) \end{cases} \tag{1..18}$$

has a solution iff $M_y - N_x = 0$.