APPENDIX A

ORDINARY DIFFERENTIAL EQUATIONS: REVIEW OF CONCEPTS AND METHODS

Among all the mathematical disciplines the theory of differential equations is the most important. It furnishes the explanation of all those elementary manifestations of nature which involve time.

                                                                    –SOPHUS LIE

Topics to Review
This appendix is self-contained and requires basic knowledge of differential and integral calculus.

Looking Ahead...
This appendix presents standard techniques for solving linear ordinary differential equations. They are needed throughout this book. Our primary use of the power series and Frobenius methods of Sections A.4, A.5, and A.6 will be to establish the basic properties of Bessel functions (Chapter 4) and Legendre polynomials (Chapter 5).

A basic knowledge of ordinary differential equations is essential for the development of the main topics of this book. For example, solving a partial differential equation by standard methods leads naturally to ordinary differential equations. In this appendix we provide a review of the basic techniques that will be useful in this book. Our brief presentation covers the essentials of linear ordinary differential equations as taught in a first course on differential equations. In the first section we present the fundamental properties of existence, uniqueness, and linear independence of solutions. The rest of the chapter is devoted to the treatment of the basic tools for solving linear ordinary differential equations, including series methods. This chapter is not intended as a comprehensive treatment, but rather as a convenient reference for topics that are needed in the book.

A.1   Linear Ordinary Differential Equations

An ordinary differential equation is an equation that involves an unknown function y and its derivatives $y^{(j)}$ of order j. A **linear ordinary differential equation in standard form** is an equation

$$(1) \qquad y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = g(x)$$

where n is called the **order** of the equation and $p_j(x)$ and $g(x)$ are given functions of x. The expression *standard form* refers to the fact that the leading coefficient is 1. A **solution** of (1) on an interval I is any n times differentiable function $y = y(x)$ satisfying (1) on I. The equation (1) is **homogeneous** if $g(x) \equiv 0$, otherwise it is **nonhomogeneous**.

The first- and second-order equations are of particular interest to us. We can describe completely the solution of the first one, while discussion of the second will occupy the rest of this appendix.

THEOREM 1
SOLUTION OF THE
FIRST ORDER
LINEAR
DIFFERENTIAL
EQUATION

Suppose that $p(x)$ and $g(x)$ are continuous on the interval I. Then any solution of the first-order linear differential equation

$$y' + p(x)y = g(x), \quad x \text{ in } I,$$

is of the form

$$(2) \qquad y = e^{-\int p(x)\,dx}\left[C + \int g(x)e^{\int p(x)\,dx}dx\right],$$

where $\int p(x)\,dx$ represents the same antiderivative of $p(x)$ in both occurences.

The expression $\int p(x)\,dx$ represents an antiderivative of $p(x)$, and so it is defined up to a constant C. In applying (2), you should take the same value of C in both occurences of $\int p(x)\,dx$. Usually, the most convenient choice is $C = 0$. (See Example 1.)

Proof Suppose y is a solution. Multiply the equation through by the **integrating factor** $\mu(x) = e^{\int p(x)\,dx}$ and get

$$\mu(x)\left[y' + p(x)y\right] = g(x)\mu(x).$$

Since $\mu'(x) = p(x)\mu(x)$, by the product rule, the equation can be rewritten as $[\mu(x)y]' = g(x)\mu(x)$. Integrating both sides and dividing by $\mu(x)$ yields the desired solution. (Note that, since $\mu(x)$ is an exponential function, it is nonzero for all x, and so we may divide by it.) ∎

EXAMPLE 1   A first-order differential equation
Using Theorem 1, we find all solutions of the differential equation

$$y' - y = 2.$$

The integrating factor is $\mu(x) = e^{-\int dx} = e^{-x}$ and hence the general solution is $y = e^x[C - 2e^{-x}] = Ce^x - 2$, where C is an arbitrary constant. ∎

In treating higher-order equations we are not so fortunate, in that, in general, we do not have a closed form for the solutions. However, there are fundamental results concerning the general form of the solutions that we discuss in the remainder of this section. Important cases where solutions are given explicitly are presented in Sections A.2 and A.3. Our discussion focuses first on the homogeneous equation

$$(3) \qquad y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0,$$

where all the coefficient functions are continuous on some fixed interval I.

We begin with a definition that is central to our treatment.

THE WRONSKIAN

Let $y_1, y_2, \ldots, y_n$ be any n solutions to the homogeneous linear differential equation (3). The **Wronskian** $W(y_1, y_2, \ldots, y_n)$ of these solutions is given by the following $n \times n$ determinant:

$$W(y_1, y_2, \ldots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \vdots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}.$$

We will sometimes use the alternative notation $W(y_1, y_2, \ldots, y_n)(x)$, or simply $W(x)$, to emphasize that the Wronskian is a function of x.

EXAMPLE 2   Computing Wronskians

(a) The equation $y'' - y = 0$ has solutions $y_1 = e^x$ and $y_2 = e^{-x}$. Their Wronskian is

$$W(e^x, e^{-x}) = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = e^x(-e^{-x}) - e^x(e^{-x}) = -2.$$

(b) The equation $y'' + y = 0$ has solutions $y_1 = \cos x$ and $y_2 = \sin x$. Their Wronskian is

$$W(\cos x, \sin x) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1.$$

■

The next theorem gives an explicit form of the Wronskian in terms of the coefficient functions in (3).

THEOREM 2
ABEL'S FORMULA
FOR THE
WRONSKIAN

Let $y_1, y_2, \ldots, y_n$ be any n solutions to the homogeneous linear differential equation (3). Then the Wronskian $W(x)$ satisfies the first-order differential equation

$$W'(x) + p_{n-1}(x)W(x) = 0, \quad \text{for } x \text{ in } I,$$

and hence

(4) $$W(x) = Ce^{-\int p_{n-1}(x)\,dx}.$$

Consequently, either $W(x) \neq 0$ for all x in I, or $W(x) \equiv 0$ on I.

The point of this theorem is that determining that $W(x) \neq 0$ for some x in I allows us to conclude that $W(x) \neq 0$ for all x in I.

Proof For clarity's sake, we give a proof only for $n = 2$. (The case $n = 3$ is treated in Exercise 31.) The same approach generalizes to higher dimensions but requires a greater knowledge of linear algebra. For $n = 2$, the equation is $y'' + p_1(x)y' + p_0(x)y = 0$, and $W(x) = y_1 y_2' - y_1' y_2$. Since y_1 and y_2 are solutions, we have

Since the key issue is whether W vanishes or not, we cannot divide by W to separate variables. An appeal to **Theorem** 1 is necessary.

$$
\begin{aligned}
W'(x) &= y_1 y_2'' - y_1'' y_2 \\
&= y_1[-p_1(x)y_2' - p_0(x)y_2] - [-p_1(x)y_1' - p_0(x)y_1]y_2 \\
&= -p_1(x)W(x).
\end{aligned}
$$

This is a first-order differential equation for $W(x)$, and (4) is an immediate consequence of (2) in Theorem 1. ∎

It is crucial in applying Theorem 2 to put the equation in standard form and verify the continuity of the coefficients. See Exercise 23.

We can now state a fundamental result in the theory of ordinary differential equations.

THEOREM 3
EXISTENCE OF
FUNDAMENTAL
SETS OF SOLUTIONS

The homogeneous equation (3)

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0,$$

where the coefficient functions $p_j(x)$ are all continuous on an interval I, has n solutions $y_1, y_2, \ldots, y_n$ with nonvanishing Wronskian on I. Furthermore, given any such set $y_1, y_2, \ldots, y_n$ and any solution y, then $y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$ for a unique choice of constants $c_1, c_2, \ldots, c_n$. The set of solutions $y_1, y_2, \ldots, y_n$ with nonvanishing Wronskian is called a **fundamental set of solutions**.

Theorem 3 asserts that any linear homogeneous differential equation of order n has n solutions that *span* the set of all solutions of the equation. Note that the theorem does not assert the uniqueness of a fundamental set. If $y_1, y_2, \ldots, y_n$ is a fundamental set of solutions, then the **linear combination** $y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$ is referred to as the **general solution**.

We have been using without verification that a linear combination of solutions of a homogeneous linear equation is again a solution. This simple fact. called the superposition principle, can be checked directly by appealing to (3). We state it here for ease of reference.

THEOREM 4 **SUPERPOSITION** **PRINCIPLE**	Suppose that $u(x)$ and $v(x)$ are solutions of the linear homogeneous equation (3) and let c and d be any two numbers. Then the linear combination $c\,u(x) + d\,v(x)$ is also a solution of (3).

Nonhomogeneous Differential Equations

Let us recall from (1) the general nonhomogeneous equation

$$(5) \qquad y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = g(x),$$

where the coefficient functions and $g(x)$ are all assumed to be continuous on some interval I. If $g(x)$ is replaced by 0, we call the resulting equation the **associated homogeneous equation**. We know from Theorem 3 that the associated homogeneous equation has a fundamental set of solutions y_1, y_2, ..., y_n. We will show in Section A.2, using the method of variation of parameters, that (5) has at least one solution that we will denote by y_p and call a **particular solution**. Now suppose that y is any other solution of (5). It is a simple exercise to check that $y - y_p$ is a solution of the associated homogeneous equation. By Theorem 3 we must have $y - y_p = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$. Thus. if y_h denotes the general solution of the associated homogeneous equation, it follows that $y = y_h + y_p$ (known as the **general solution** of the nonhomogeneous equation). We have proved the following important result.

THEOREM 5 **GENERAL** **SOLUTION OF NON-** **HOMOGENEOUS** **EQUATIONS**	Let y_h and y_p denote, respectively, the general solution of the homogeneous equation associated with (5) and a particular solution of (5). Then any solution y of (5) has the form $$y = y_h + y_p.$$

Often in applications we have to solve differential equations subject to certain conditions; for example,

$$y'' + y = e^x, \qquad y(0) = 0,\ y'(0) = 1.$$

Such a problem is called an **initial value problem**, and the conditions are called **initial conditions**. Typically we impose enough conditions to specify a unique solution of the problem. Since the general solution of an nth order linear differential equation has n arbitrary constants, we expect n conditions to suffice.

THEOREM 6
EXISTENCE AND
UNIQUENESS FOR
INITIAL VALUE
PROBLEMS

Consider the initial value problem consisting of the linear differential equation (5) and the initial conditions

$$y(x_0) = y_0, \ y'(x_0) = y_0', \ \ldots, \ y^{(n-1)}(x_0) = y_0^{(n-1)},$$

where x_0 is in I and y_0, y_0', $\ldots$, $y_0^{(n-1)}$ are prescribed values. Then this problem has a unique solution y on the interval I.

Proof Let y_p be a particular solution to (5) and let $y_1, y_2, \ldots, y_n$ be a fundamental set of solutions to the associated homogeneous equation. For clarity's sake we take $n = 2$, although the proof generalizes for arbitrary n. We need to solve the 2×2 system

$$\begin{cases} c_1 y_1(x_0) + c_2 y_2(x_0) + y_p(x_0) = y_0 \\ c_1 y_1'(x_0) + c_2 y_2'(x_0) + y_p'(x_0) = y_0' \end{cases} \Leftrightarrow \begin{cases} c_1 y_1(x_0) + c_2 y_2(x_0) = y_0 - y_p(x_0) \\ c_1 y_1'(x_0) + c_2 y_2'(x_0) = y_0' - y_p'(x_0) \end{cases}$$

Figure 1 Various solutions from Example 3.

for the unknowns c_1 and c_2. Since the determinant of the matrix of this system is precisely the Wronskian and is nonzero by definition (see Theorem 3), it follows that the system has a unique solution pair c_1, c_2. This proves the existence of the solution of the initial value problem. To establish the uniqueness of the solution, suppose that $u(x)$ and $v(x)$ are two solutions to the initial value problem. Let $w = u - v$. Then w satisfies the associated homogeneous equation with 0 initial values. We know from Theorem 3 that $w = ay_1 + by_2$ for some choice of a and b. Now using the initial values of w and the fact that $W(y_1, y_2)(x_0) \neq 0$, we infer that $a = b = 0$ and hence $w \equiv 0$ implying that $u \equiv v$. ∎
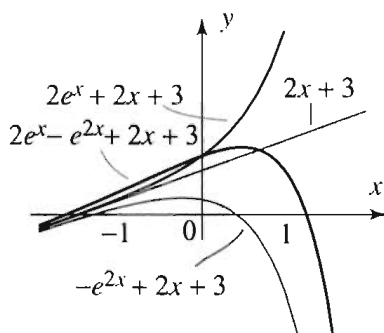

EXAMPLE 3   An initial value problem

You can (and should) check that the equation $y'' - 3y' + 2y = 4x$ has $y_p = 2x + 3$ as a particular solution and that the associated homogeneous equation $y'' - 3y' + 2y = 0$ has general solution $y_h = c_1 e^x + c_2 e^{2x}$. (General techniques for deriving these solutions will be developed in the next two sections.) Hence the general solution of the nonhomogeneous equation is

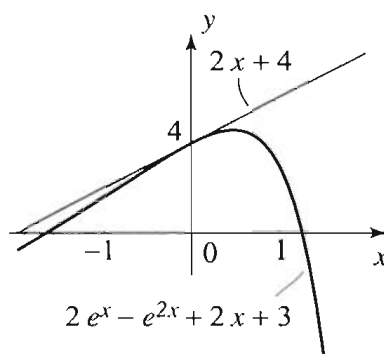$$y = y_h + y_p = c_1 e^x + c_2 e^{2x} + 2x + 3$$

Figure 2 Solution of the initial value problem in Example 3 and its tangent line at $x = 0$. It is the only curve among those in Figure 1 that goes through the point $(0, 4)$ and that has slope at $x = 0$ equal to $y'(0) = 2$.

(see Figure 1). Now suppose that we want to solve the initial value problem

$$y'' - 3y' + 2y = 4x, \quad y(0) = 4, \ y'(0) = 2.$$

Using the initial conditions, we get

$$y(0) = 4 \Rightarrow c_1 + c_2 = 1,$$
$$y'(0) = 2 \Rightarrow c_1 + 2c_2 = 0.$$

This system has a unique solution $c_1 = 2$ and $c_2 = -1$. Hence the (unique) solution of the initial value problem is $y = 2e^x - e^{2x} + 2x + 3$ (Figure 2). ∎

We next present a brief discussion of linear independence of solutions and its connection to fundamental sets.

The functions f_1, f_2, ..., f_n defined on an interval $[a, b]$ are said to be **linearly independent** on $[a, b]$ if the only choice of the constants c_1, c_2, ..., c_n for which the linear combination $c_1 f_1 + c_2 f_2 + \cdots + c_n f_n$ vanishes identically on $[a, b]$ is $c_1 = c_2 = \cdots = c_n = 0$. Otherwise, the functions are said to be **linearly dependent**. When $n = 2$, we note that linear dependence is equivalent to one function being a constant multiple of the other. The following result shows a connection between the notion of linear independence and fundamental sets.

THEOREM 7
WRONSKIAN
CRITERION FOR
LINEAR
INDEPENDENCE

Let $y_1, y_2, \ldots, y_n$ be any n solutions to the homogeneous linear differential equation (3) with coefficients continuous on an interval I. The following are equivalent:

(i) $y_1, y_2, \ldots, y_n$ are linearly independent on I;
(ii) $y_1, y_2, \ldots, y_n$ form a fundamental set of solutions of (3);
(iii) $W(y_1, y_2, \ldots, y_n)(x_0) \neq 0$ for some x_0 in I;
(iv) $W(y_1, y_2, \ldots, y_n)(x) \neq 0$ for all x in I.

Proof Clearly (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) follow from Theorems 2 and 3 and the definition of a fundamental set.   We conclude by proving (i) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (i). We start with (iii) $\Rightarrow$ (i). Suppose that $y_1, y_2, \ldots, y_n$ are any n solutions such that $c_1 y_1 + c_2 y_2 + \cdots + c_n y_n = 0$. By differentiating this equation $n - 1$ times in succession and then setting $x = x_0$, we get a system of n linear homogeneous equations in the n unknowns $c_1, c_2, \ldots, c_n$.   The determinant of the matrix of this system is precisely $W(x_0)$, which is nonzero by assumption.   The nonvanishing of this determinant implies that $c_1 = c_2 = \cdots = c_n = 0$, proving (iii) $\Rightarrow$ (i). To complete the proof it is enough to show (i) $\Rightarrow$ (iii). The proof is by contradiction. We assume that $y_1, y_2, \ldots, y_n$ are linearly independent on I and that $W(y_1, y_2, \ldots, y_n)(x_0) = 0$. The vanishing of the Wronskian implies that there are constants $c_1, c_2, \ldots, c_n$, not all zero, such that all the initial values of $y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$ are 0 at x_0 (for $n = 2$, **this fact is** clear). However, the solution $w \equiv 0$ has the same initial values, and so **by the uniq**ueness part of Theorem 6, $w \equiv y \equiv 0$, implying that $c_1 = c_2 = \cdots = c_n = 0$ by the definition of linear independence, which is a contradiction.   ∎

EXAMPLE 4   Fundamental sets of solutions, linear independence
We saw in Example 2 that e^x and e^{-x} are solutions of $y'' - y = 0$ and that $W(e^x, e^{-x}) = -2 \neq 0$.   Thus, by Theorem 7, the general solution of the differential equation is of the form $y = c_1 e^x + c_2 e^{-x}$; and so any solution is a linear combination of the functions e^x and e^{-x}, **which form** a fundamental set of solutions.

It is worthwhile to note that the fundamental set of solutions is not unique. For the equation at hand, you can easily check that $\cosh x$ and $\sinh x$ are also solutions. Their Wronskian is

$$W(\cosh x, \sinh x) = \begin{vmatrix} \cosh x & \sinh x \\ \sinh x & \cosh x \end{vmatrix} = \cosh^2 x - \sinh^2 x = 1.$$

Thus, by Theorem 7, the general solution of the differential equation is also of the form $y = c_1 \cosh x + c_2 \sinh x$. ∎

Thus far we have developed an understanding of the general form of the solutions of linear differential equations. However, aside from the first order case, we have not developed techniques for finding explicit solutions. This will occupy us throughout the remainder of this appendix, where we will develop methods for solving certain important classes of higher order linear differential equations.

Exercises A.1

In Exercises 1–10, solve the given first order differential equation.

1. $y' + y = 1$. **2.** $y' + 2xy = x$. **3.** $y' = -.5\,y$.

4. $y' = 2y + x$. **5.** $y' - y = \sin x$. **6.** $y' - 2x\,y = x^3$.

7. $xy' + y = \cos x$. **8.** $y' - \frac{2}{x}y = x^2$. **9.** $y' + \tan x\,y = \cos x$.

10. $y' + \tan x\,y = \sec^2 x$.

In Exercises 11–20, solve the given initial value problem.

11. $y' = y$, $y(0) = 1$. **12.** $y' + 2y = 1$, $y(0) = 2$.

13. $y' + xy = x$, $y(0) = 0$. **14.** $y' + \frac{y}{x} = \frac{\sin x}{x}$, $y(\pi) = 1$.

15. $xy' + 2y = 1$, $y(-1) = -2$. **16.** $xy' - 2y = \frac{1}{x}$, $y(1) = 0$.

17. $y' + y\tan x = \tan x$, $y(0) = 1$. **18.** $y' + y\tan x = \tan x$, $y(0) = 2$.

19. $y' + e^x y = e^x$, $y(0) = 2$. **20.** $y' + y = e^x$, $y(3) = 0$.

21. (a) Check that the functions

$$e^x,\ e^{-x}, \cosh x,\ \sinh x$$

are solutions of

$$y'' - y = 0.$$

(b) We know from Example 4 that $\{e^x, e^{-x}\}$ and $\{\cosh x, \sinh x\}$ are fundamental sets of solutions. Express e^x as a linear combination of the functions $\cosh x$, $\sinh x$.
(c) Can you think of other fundamental sets of solutions?

22. Check that the functions e^x, e^{-x}, $\cosh x$ are solutions of

$$y''' - y' = 0.$$

(b) Show directly that $W(e^x, e^{-x}, \cosh x) \equiv 0$ and conclude that these functions do not form a fundamental set.
(c) Find a fundamental set of solutions by inspection. Justify your answer by computing the Wronskian.

23. (a) Check that the functions x, x^2 are solutions of

$$x^2 y'' - 2xy' + 2y = 0.$$

(b) Compute the Wronskian of the solutions.

(c) Note that the Wronskian vanishes at $x = 0$. Does this contradict Theorem 2?

24. (a) Check that the functions e^x, $1 + x$ are solutions of

$$xy'' - (1 + x)y' + y = 0.$$

(b) Check that both of these functions satisfy $y(0) = 1$ and $y'(0) = 1$. Does this contradict the uniqueness part in Theorem 6?

(c) Compute the Wronskian of the solutions and conclude that they are linearly independent on $(0, \infty)$.

In Exercises 25–30, solve the initial value problem consisting of the differential equation in Example 3.

25. $y(0) = 0$, $y'(0) = 0$.   **26.** $y(0) = 1$, $y'(0) = -1$.

27. $y(1) = 0$, $y'(1) = 2$.   **28.** $y(1) = 1$, $y'(1) = -1$.
[Hint: In Exercises 27 and 28, it is easier to work with $y_h = c_1 e^{(x-1)} + c_2 e^{2(x-1)}$. Why is this possible?]

29. $y(2) = 0$, $y'(2) = 1$.   **30.** $y(3) = 3$, $y'(3) = 3$.
[Hint: In Exercises 29–30, use a y_h that simplifies the computations as in the previous exercises.]

31. Project Problem: Abel's formula for $n = 3$.

(a) Let y_1, y_2, y_3 be any three solutions of the third order equation

$$y''' + p_2(x)y'' + p_1(x)y' + p_0(x)y = 0.$$

Derive the formulas for the Wronskian

$$W(y_1, y_2, y_3) = (y_2 y_3' - y_2' y_3)y_1'' - (y_1 y_3' - y_1' y_3)y_2'' + (y_1 y_2' - y_1' y_2)y_3''$$

and

$$W'(y_1, y_2, y_3) = (y_2 y_3' - y_2' y_3)y_1''' - (y_1 y_3' - y_1' y_3)y_2''' + (y_1 y_2' - y_1' y_2)y_3'''.$$

(b) Follow the proof of Theorem 2 to show that

$$W' = -p_2(x)W.$$

(c) Derive Abel's formula for $n = 3$.

32. (a) Show that $W(x^3, |x^3|) = 0$ for all x. (The function $|x^3|$ is differentiable for all x. Can you see why?)

(b) Show, however, that x^3, $|x^3|$ are linearly independent on the real line.

(c) Does this contradict Theorem 7? Justify your answer.

A.2   Linear Ordinary Differential Equations with Constant Coefficients

The general form of the **nth order homogeneous linear differential equation with constant coefficients** is

(1) $$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0,$$

where each a_j is a constant and $a_n \neq 0$. The equation can be put in standard form by dividing through by the leading coefficient a_n. This important class of equations is needed throughout this book. Our presentation will emphasize the second-order case, which is by far the most useful.

The simple first-order case of (1) suggests that we try

$$y = e^{\lambda x}$$

in solving the general case. Since

$$y' = \lambda e^{\lambda x}, \ y'' = \lambda^2 e^{\lambda x}, \ \ldots, \ y^{(n-1)} = \lambda^{n-1} e^{\lambda x}, \ y^{(n)} = \lambda^n e^{\lambda x},$$

substituting these into (1), we get

$$(a_n \lambda^n + a_{n-1} \lambda^{n-1} + \cdots + a_1 \lambda + a_0) e^{\lambda x} = 0.$$

Thus $y = e^{\lambda x}$ is a solution to (1) if λ is any root of the **characteristic equation**

$$\boxed{a_n \lambda^n + a_{n-1} \lambda^{n-1} + \cdots + a_1 \lambda + a_0 = 0.}$$

The roots of this equation are called the **characteristic roots**. For future use, we define the **characteristic polynomial**

$$p(\lambda) = a_n \lambda^n + a_{n-1} \lambda^{n-1} + \cdots + a_1 \lambda + a_0.$$

In the case when this polynomial has n distinct characteristic roots λ_1, λ_2, ..., λ_n, the general solution of (1) is

$$\boxed{y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} + \cdots + c_n e^{\lambda_n x}.}$$

Equivalently, a fundamental set of solutions is given by

(2) $$\boxed{e^{\lambda_1 x}, \ e^{\lambda_2 x}, \ \ldots, \ e^{\lambda_n x}.}$$

This follows from the nonvanishing of the Wronskian $W(e^{\lambda_1 x}, e^{\lambda_2 x}, \ldots, e^{\lambda_n x})$. See Exercise 77.
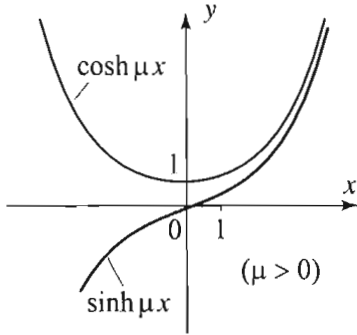
EXAMPLE 1   Hyperbolic functions

The equation $y'' - \mu^2 y = 0$ $(\mu > 0)$ occurs frequently in this book. Its characteristic equation is

$$\lambda^2 - \mu^2 = 0.$$

with characteristic roots $\lambda_1 = \mu$ and $\lambda_2 = -\mu$. Thus a fundamental set of solutions is $\{e^{\mu x}, e^{-\mu x}\}$. As we observed following Example 4 of the previous section, the fundamental set of solutions is not unique. In fact, we now describe another set that is more convenient in some applications. Recall the definition of the hyperbolic functions

$$\cosh \mu x = \frac{e^{\mu x} + e^{-\mu x}}{2} \quad \text{and} \quad \sinh \mu x = \frac{e^{\mu x} - e^{-\mu x}}{2}.$$

Since these are linear combinations of functions from the fundamental set of solutions, they are themselves solutions of the differential equation. You could also verify the last assertion directly by using the derivative formulas

$$\frac{d}{dx} \cosh \mu x = \mu \sinh \mu x \quad \text{and} \quad \frac{d}{dx} \sinh \mu x = \mu \cosh \mu x.$$

Computing the Wronskian of $\cosh \mu x$ and $\sinh \mu x$, we find

$$W(\cosh \mu x, \sinh \mu x) = \begin{vmatrix} \cosh \mu x & \sinh \mu x \\ \mu \sinh \mu x & \mu \cosh \mu x \end{vmatrix} = \mu \overbrace{(\cosh^2 \mu x - \sinh^2 \mu x)}^{=1} = \mu \neq 0.$$

By Theorem 7 of Appendix A.1 (with $n = 2$), we conclude that $\cosh \mu x$ and $\sinh \mu x$ form a fundamental set of solutions and so the general solution of the differential equation is of the form $y = c_1 \cosh \mu x + c_2 \sinh \mu x$. Basic properties of $\cosh \mu x$ and $\sinh \mu x$ are illustrated by the graphs shown in Figure 1. ∎

Figure 1 The hyperbolic cosine and sine (with $\mu > 0$): $\cosh \mu x$ is even and $\cosh \mu x \geq 1$ for all x.
$\sinh \mu x$ is odd and $\sinh \mu x = 0$ if and only if $x = 0$.

EXAMPLE 2   Characteristic equation with distinct real roots

The third-order equation $y''' - y'' - 6y' = 0$ has characteristic equation

$$\lambda^3 - \lambda^2 - 6\lambda = 0.$$

Since this equation factors as $\lambda(\lambda - 3)(\lambda + 2) = 0$, we get the characteristic roots $\lambda_1 = 0, \lambda_2 = 3, \lambda_3 = -2$. Thus the general solution of the differential equation is $y = c_1 + c_2 e^{3x} + c_3 e^{-2x}$. Observe that $\{1, e^{3x}, e^{-2x}\}$ is a fundamental set of solutions. You should check that $W(1, e^{3x}, e^{-2x}) \neq 0$. ∎
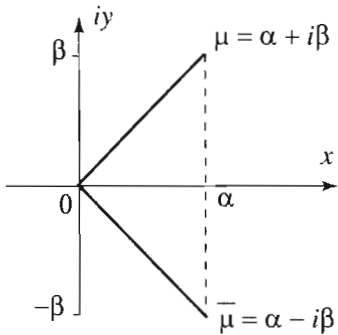
In the case of n distinct roots, the functions in (2) can still be used as a fundamental set even when some or all of the characteristic roots are complex numbers. However, for practical purposes, it is preferable to have real-valued functions. This can be done by using the fact that for any nonreal root $\mu = \alpha + i\beta$ $(\beta \neq 0)$ of a polynomial with real coefficients its complex conjugate $\overline{\mu} = \alpha - i\beta$ is also a root (Figure 2). We illustrate this technique with a simple but very important example.

Figure 2 The complex number $\mu = \alpha + i\beta$ and its conjugate $\overline{\mu} = \alpha - i\beta$.

EXAMPLE 3   Trigonometric functions

The characteristic equation of $y'' + k^2 y = 0 \ (k > 0)$ is

$$\lambda^2 + k^2 = 0.$$

Its characteristic roots are $\lambda_1 = ik$ and $\lambda_2 = -ik$. Thus a fundamental set of solutions is $\{e^{ikx},\ e^{-ikx}\}$. We now describe an alternative fundamental set of real-valued solutions. The construction is similar to the one we used in constructing the hyperbolic cosine and sine solutions in the previous example. Recall Euler's identity: $e^{ikx} = \cos kx + i \sin kx$. Consider the following solution, which is formed by taking a particular linear combination from the fundamental set of solutions:

$$
\begin{aligned}
y_1 &= \frac{e^{ikx} + e^{-ikx}}{2} = \frac{1}{2}\left(\cos kx + i \sin kx + \cos kx - i \sin kx \right) \\
&= \cos kx.
\end{aligned}
$$

Thus $y_1 = \cos kx$ is a solution of the differential equation, which is a fact that you can verify directly. Similarly, the linear combination $\left(e^{ikx} - e^{-ikx}\right)/2i$ yields the solution $y_2 = \sin kx$. Clearly, y_1 and y_2 are real-valued and linearly independent (check their Wronskian). Thus the general solution of $y'' + k^2 y = 0 \ (k > 0)$ is $y = c_1 \cos kx + c_2 \sin kx$. ∎

In the previous example, the characteristic roots were purely imaginary (their real parts $= 0$). The next example illustrates a case in which the characteristic roots have nonzero real parts. The solutions will involve real exponential functions, along with cosines and sines.

EXAMPLE 4   Complex characteristic roots

The characteristic roots of the equation $y'' - 4y' + 5y = 0$ are $\lambda_1 = 2 + i$ and $\lambda_2 = 2 - i$. From (2), the functions $e^{(2+i)x}$ and $e^{(2-i)x}$ form a fundamental set of solutions. We next show how to obtain real-valued solutions. Using Euler's identity, $e^{i\theta} = \cos \theta + i \sin \theta$, we have

$$e^{(2+i)x} = e^{2x} e^{ix} = e^{2x}(\cos x + i \sin x).$$

Similarly, $e^{(2-i)x} = e^{2x}(\cos x - i \sin x)$. By taking sums and differences of these solutions, we arrive at the solutions $2 e^{2x} \cos x$ and $2i e^{2x} \sin x$. Thus $e^{2x} \cos x$ and $e^{2x} \sin x$ are two real-valued solutions. We have

$$
\begin{aligned}
W(e^{2x} \cos x, e^{2x} \sin x) &= \begin{vmatrix} e^{2x} \cos x & e^{2x} \sin x \\ e^{2x}(2 \cos x - \sin x) & e^{2x}(2 \sin x + \cos x) \end{vmatrix} \\
&= e^{4x}\left((2 \sin x + \cos x) \cos x - (2 \cos x - \sin x) \sin x \right) \\
&= e^{4x}(\cos^2 x + \sin^2 x) = e^{4x} \neq 0.
\end{aligned}
$$

We conclude that $e^{2x} \cos x$ and $e^{2x} \sin x$ form a fundamental set of solutions. Note that in the case of two solutions, we could have verified that the set is fundamental by simply observing that one is not a constant multiple of the other. ∎

In general, when we have complex characteristic roots, these roots come in conjugate pairs. A fundamental set of real-valued solutions is obtained by replacing the pair of functions $e^{\mu x}$, $e^{\bar{\mu} x}$ in (2) by

(3)
$$\boxed{e^{\alpha x} \cos \beta x, \; e^{\alpha x} \sin \beta x}$$

for each nonreal pair of characteristic roots $\mu = \alpha \pm i\beta$. Let us illustrate the pairing of complex roots with one more example.

EXAMPLE 5   Pairing complex characteristic roots

Find a fundamental set of solutions of $y''' - y = 0$.

Solution Let us find the characteristic roots:

$$\lambda^3 - 1 = 0 \quad \Rightarrow \quad (\lambda - 1)(\lambda^2 + \lambda + 1) = 0$$

$$\Rightarrow \quad \lambda_1 = 1, \; \lambda_2 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \; \lambda_3 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

We have one real root and two complex conjugate roots. To the real root corresponds the solution $y_1 = e^x$, and to the pair of complex conjugate roots correspond the two solutions $y_2 = e^{-x/2} \cos \left(\frac{\sqrt{3}}{2} x \right)$ and $y_3 = e^{-x/2} \sin \left(\frac{\sqrt{3}}{2} x \right)$. The functions y_1, y_2, and y_3 form a fundamental set of solutions. ∎

Thus far we have not dealt with the case of a repeated root μ of the characteristic polynomial of multiplicity $m > 1$. If we try to use (2), we get the function $e^{\mu x}$ repeated m times, and hence we do not obtain a fundamental set. The following example illustrates how to resolve this problem.

EXAMPLE 6   Repeated roots of the characteristic polynomial

(a) The second-order equation $y'' + 2y' + y = 0$ has characteristic polynomial $(\lambda+1)^2$, which has -1 as a root of multiplicity two. One solution of the equation is e^{-x}. A simple verification shows that the function xe^{-x} is also a solution of the differential equation, which is also independent of e^{-x}. We conclude that e^{-x} and xe^{-x} form a fundamental set of solutions (Figure 3). (We note that the method of reduction of order, presented in the next section, provides a straightforward way to obtain the second solution xe^{-x}.)

(b) The third-order equation $y''' - 6y'' + 12y' - 8y = 0$ has characteristic polynomial $(\lambda - 2)^3$, which has 2 as a root of multiplicity three. One solution of the equation is e^{2x}. Taking a hint from the previous example, we try $y = xe^{2x}$ for a second solution. We have

$$y' = e^{2x}(1 + 2x), \quad y'' = 4e^{2x}(1 + x), \quad y''' = 4e^{2x}(3 + 2x).$$

Plugging into the equation, we obtain

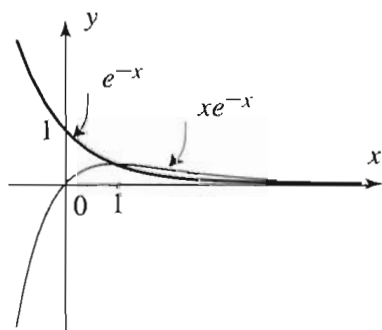$$4e^{2x}(3 + 2x) - 24e^{2x}(1 + x) + 12e^{2x}(1 + 2x) - 8xe^{2x} = 0.$$

Figure 3 Solutions from Example 6(a).

Thus $y = xe^{2x}$ is a solution. For a third solution of the differential equation, we modify the first solution by multiplying by x^2, and try $y = x^2 e^{2x}$. You can check that

$$y' = 2xe^{2x}(1+x), \quad y'' = 2e^{2x}(1+4x+2x^2), \quad y''' = 4e^{2x}(3+6x+2x^2),$$

and that $x^2 e^{2x}$ is a solution. It is easy to check that the solutions $e^{2x}, xe^{2x}, x^2 e^{2x}$ are linearly independent, and hence they form a fundamental set of solutions of the differential equation. ∎

This example motivates the following prescription for obtaining a fundamental set of solutions of (1) in the case when multiple roots occur. Let μ be a repeated root having multiplicity m. In (2), we replace the m occurrences of $e^{\mu x}$ by the following m functions:

Note that there are m linearly independent solutions corresponding to repeated roots of order m.

(4)
$$\boxed{e^{\mu x}, \ xe^{\mu x}, \ \ldots, \ x^{m-1}e^{\mu x}.}$$

To obtain a fundamental set of solutions of (1), make this replacement for each repeated characteristic root. (Keep in mind that the multiplicity m may vary from one root to another.)

Finally, if $\mu = \alpha + i\beta$ is a nonreal repeated characteristic root of multiplicity m, to get a fundamental set of real-valued solutions we combine the previous methods and find that the $2m$ functions associated with the roots μ and $\overline{\mu}$ are

(5)
$$e^{\alpha x}\cos\beta x, \ xe^{\alpha x}\cos\beta x, \ \ldots, \ x^{m-1}e^{\alpha x}\cos\beta x;$$
$$e^{\alpha x}\sin\beta x, \ xe^{\alpha x}\sin\beta x, \ \ldots, \ x^{m-1}e^{\alpha x}\sin\beta x.$$

For ease of reference, we restate in the following box the results of our discussion in terms of general solutions of (1). For convenience we view a nonrepeated characteristic root as a root of multiplicity $m = 1$.

GENERAL SOLUTION OF THE nth-ORDER LINEAR HOMOGENEOUS EQUATION WITH CONSTANT COEFFICIENTS

Consider the equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

with characteristic equation

$$a_n \lambda^n + a_{n-1} \lambda^{n-1} + \cdots + a_1 \lambda + a_0 = 0.$$

The general solution of the differential equation is constructed according to the nature of the roots of the characteristic equation as follows.

Case I   For each real root μ of multiplicity $m \geq 1$, we include a linear combination of the form

$$c_1 e^{\mu x} + c_2 x e^{\mu x} + \cdots + c_m x^{m-1} e^{\mu x}.$$

Case II   For each nonreal root $\mu = \alpha + i\beta$ and its complex conjugate $\bar{\mu} = \alpha - i\beta$ of multiplicity $m \geq 1$, we include a linear combination of the form

$$c_1 e^{\alpha x} \cos \beta x + c_2 x e^{\alpha x} \cos \beta x + \cdots + c_m x^{m-1} e^{\alpha x} \cos \beta x$$

$$+ d_1 e^{\alpha x} \sin \beta x + d_2 x e^{\alpha x} \sin \beta x + \cdots + d_m x^{m-1} e^{\alpha x} \sin \beta x.$$

EXAMPLE 7   Fourth-order equations

Find the general solution of the following equations:
(a) $y^{(4)} - 16 y = 0$;
(b) $y^{(4)} + 2y'' + y = 0$.

Solution   (a) The characteristic equation is $\lambda^4 - 16 = 0$, with characteristic roots ± 2 and $\pm 2i$. Thus, the general solution is

$$y = c_1 e^{2x} + c_2 e^{-2x} + d_1 \cos 2x + d_2 \sin 2x.$$

(b) The characteristic equation is

$$(\lambda^2 + 1)^2 = 0 \quad \text{or} \quad (\lambda - i)^2 (\lambda + i)^2 = 0.$$

Thus the characteristic roots are i and $-i$ with multiplicity 2 each. Accordingly, the general solution is

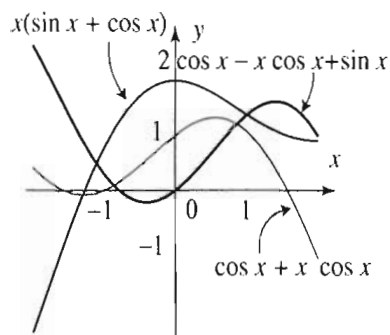$$y = c_1 \cos x + c_2 x \cos x + d_1 \sin x + d_2 x \sin x.$$

Figure 4 shows some specific solutions that are obtained by assigning different values to the constants c_1, c_2, d_1, d_2.   ∎

Figure 4 Various solutions from Example 7(b).

In the second-order case, we have a simpler form of the general solution which we now describe. You will be asked to verify these solutions in the exercises.

GENERAL SOLUTION OF THE SECOND-ORDER LINEAR HOMOGENEOUS EQUATION WITH CONSTANT COEFFICIENTS

Consider the equation

$$(6) \qquad ay'' + by' + cy = 0$$

with characteristic equation

$$a\lambda^2 + b\lambda + c = 0.$$

Let λ_1 and λ_2 denote the characteristic roots. The general solution y of this differential equation is given by one of the following cases.

Case I If λ_1 and λ_2 are distinct real roots, then

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}.$$

Equivalently, write $\lambda_1 = \frac{-b}{2a} + \frac{\sqrt{b^2-4ac}}{2a} = \alpha + \beta$ and $\lambda_2 = \alpha - \beta$, then

$$y = e^{\alpha x}(c_1 \cosh \beta x + c_2 \sinh \beta x).$$

Case II If $\lambda_1 = \lambda_2$, then

$$y = c_1 e^{\lambda_1 x} + c_2 x e^{\lambda_1 x}.$$

Case III If λ_1 and λ_2 are complex conjugate roots with $\lambda_1 = \alpha + i\beta$, then

$$y = c_1 e^{\alpha x} \cos \beta x + c_2 e^{\alpha x} \sin \beta x.$$

We now turn our attention to nonhomogeneous second-order equations with constant coefficients.

The Method of Undetermined Coefficients

The study of various physical systems often leads to the equation

$$(7) \qquad ay'' + by' + cy = g(x).$$

The general solution of the associated homogeneous equation y_h has already been given explicitly. Thus, to find the general solution of (7), it is enough by Theorem 5 of the previous section to find a particular solution y_p. Finding y_p depends on the nonhomogeneous term g. In many interesting cases, the form of g allows us to guess the form of y_p up to a set of unknown coefficients. This method is called the method of **undetermined coefficients**. We illustrate it with examples. We also note that in other cases where the method of undetermined coefficients does not work, we can use variation of parameters, as described in the next section.

EXAMPLE 8   A simple undetermined coefficients problem

Find the general solution of the given nonhomogeneous equation.
(a) $y'' - 4y' + 5y = e^x$.   (b) $y'' + 2y' + y = e^{-x}$.

Solution (a) The associated homogeneous equation is solved in Example 4. We have $y_h = c_1 e^{2x} \cos x + c_2 e^{2x} \sin x$. To determine the general solution, it remains to find y_p. Since the right side of the equation is e^x it makes sense to try $y_p = A e^x$, where A is an unknown constant yet to be determined. A computation shows that $y_p'' - 4y_p' + 5y_p = 2A e^x$. For y_p to be a solution we must set $A = \frac{1}{2}$. Thus the general solution is $y = c_1 e^{2x} \cos x + c_2 e^{2x} \sin x + \frac{1}{2} e^x$.

(b) The associated homogeneous equation is solved in Example 6. We have $y_h = c_1 e^{-x} + c_2 x e^{-x}$. To find y_p it is pointless to try $A e^{-x}$ or $A x e^{-x}$ since both of these solve the homogeneous equation. We modify our guess to $y_p = A x^2 e^{-x}$. Differentiating y_p, we find

$$y_p' = 2A x e^{-x} - A x^2 e^{-x}, \qquad y_p'' = 2A e^{-x} - 4A x e^{-x} + A x^2 e^{-x}.$$

Hence $y_p'' + 2y_p' + y_p = 2A e^{-x}$. For y_p to be a solution we must choose $A = \frac{1}{2}$. Thus the general solution is $y = c_1 e^{-x} + c_2 x e^{-x} + \frac{1}{2} x^2 e^{-x}$.   ∎

The procedure we have used above is covered by the following general rules.

THE METHOD OF UNDETERMINED COEFFICIENTS

To find a particular solution of (7) when

(8) $$g(x) = (a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0) e^{\alpha x} \begin{cases} \cos \beta x \\ \sin \beta x \end{cases}$$

we use

(9) $$y_p = (A_n x^n + A_{n-1} x^{n-1} + \cdots + A_0) e^{\alpha x} \cos \beta x$$
$$+ (B_n x^n + B_{n-1} x^{n-1} + \cdots + B_0) e^{\alpha x} \sin \beta x$$

provided that no term in the expression of y_p is a solution of the associated homogeneous equation. If not, we modify this expression by multiplying by x or x^2. We use x if the characteristic polynomial of the associated homogeneous equation has distinct roots, and x^2 if it has a double root.
Superposition rule If the right side of (7) is a sum of several different functions of the form (8), we use a corresponding sum of terms of the form (9).

EXAMPLE 9   The method of undetermined coefficients

Use the method of undetermined coefficients to find the general solution of

$$y'' - 3y' + 2y = x^3.$$

Solution   It is straightforward to see that $y_h = c_1 e^x + c_2 e^{2x}$. From (9), we see that $y_p = Ax^3 + Bx^2 + Cx + D$. Note that none of these terms appears in the expression of y_h.   We have

$$
\begin{aligned}
y_p'' - 3y_p' + 2y_p &= 6Ax + 2B - 3(3Ax^2 + 2Bx + C) + 2(Ax^3 + Bx^2 + Cx + D) \\
&= 2Ax^3 + (-9A + 2B)x^2 + (6A - 6B + 2C)x + 2B - 3C + 2D.
\end{aligned}
$$

For y_p to be a solution, A, B, C, and D must satisfy

$$
\begin{array}{llll}
2A & & & = 1 \\
-9A & +2B & & = 0 \\
6A & -6B & +2C & = 0 \\
& 2B & -3C & +2D = 0
\end{array}
$$

The solution of this linear system is $A = \frac{1}{2}$, $B = \frac{9}{4}$, $C = \frac{21}{4}$, $D = \frac{45}{8}$.   Thus

$$y_p = \frac{1}{2}x^3 + \frac{9}{4}x^2 + \frac{21}{4}x + \frac{45}{8},$$

and the general solution is obtained by adding on y_h.   ■

EXAMPLE 10   The method of undetermined coefficients

Use the method of undetermined coefficients to find the general solution of

$$y'' - 3y' + 2y = e^x \cos x.$$

Solution   According to (9), we take $y_p = Ae^x \cos x + Be^x \sin x$. Since neither of these terms appears in the expression of y_h (see Example 9), there is no need for modification.   Now $y_p' = (A+B)e^x \cos x + (-A+B)e^x \sin x$ and $y_p'' = 2Be^x \cos x - 2Ae^x \sin x$, and hence

$$y_p'' - 3y_p' + 2y_p = -(A + B)e^x \cos x + (A - B)e^x \sin x.$$

For y_p to be a solution, we must solve $A + B = -1$ and $A - B = 0$.   From this we get $y_p = -\frac{1}{2}e^x \cos x - \frac{1}{2}e^x \sin x$ and hence the general solution is

$$y = c_1 e^x + c_2 e^{2x} - \frac{1}{2}e^x (\cos x + \sin x).$$ ■

EXAMPLE 11   Using the superposition rule

Consider the equation $y'' + 4y = e^{-2x} + 3\sin 2x$. It is easy to see that

$$y_h = c_1 \cos 2x + c_2 \sin 2x.$$

The term e^{-2x} requires a term Ae^{-2x} in y_p. The term $3\sin 2x$ suggests the expression $B\cos 2x + C\sin 2x$, but because these terms appear in y_h we must introduce

an extra factor of x (the characteristic polynomial has distinct roots).   Hence we take

$$y_p = A e^{2x} + B x \cos 2x + C x \sin 2x.$$

Solving for A, B, and C as before, we find $y_p = \frac{1}{8} e^{-2x} - \frac{3}{4} x \cos 2x$.   ∎

Note that even though the nonhomogeneous term $g(x)$ in the equation of Example 11 has no cosine term in it, the particular solution does have one.   This emphasizes the necessity of including both cosine and sine terms in y_p even when only one of these appears in $g(x)$.

Exercises A.2

In Exercises 1–24, find the general solution of the given equation.

1. $y'' - 4y' + 3y = 0$.   2. $y'' - y' - 6y = 0$.   3. $y'' - 5y' + 6y = 0$.

4. $2y'' - 3y' + y = 0$.   5 $y'' + 2y' + y = 0$.   6. $4y'' - 13y' + 9y = 0$.

7. $4y'' - 4y' + y = 0$.   8. $4y'' - 12y' + 9y = 0$.   9. $y'' + y = 0$.

10. $9y'' + 4y = 0$.   11. $y'' - 4y = 0$.   12. $y'' + 3y' + 3y = 0$.

13. $y'' + 4y' + 5y = 0$.   14. $y'' - 2y' + 5y = 0$.   15. $y'' + 6y' + 13y = 0$.

16. $2y'' - 6y' + 5y = 0$. 17. $y''' - 2y'' + y' = 0$.   18. $y''' - 3y'' + 2y = 0$.

19. $y^{(4)} - 2y'' + y = 0$.   20. $y^{(4)} - y = 0$.

21. $y''' - 3y'' + 3y' - y = 0$.   22. $y''' + y = 0$.

23. $y^{(4)} - 6y'' + 8y' - 3y = 0$.   24. $y^{(4)} + 4y''' + 6y'' + 4y' + y = 0$.

In Exercises 25–44, find the general solution of the given equation using the method of undetermined coefficients.

25. $y'' - 4y' + 3y = e^{2x}$.   26. $y'' - y' - 6y = e^{x}$.

27. $y'' - 5y' + 6y = e^{x} + x$.   28. $2y'' - 3y' + y = e^{2x} + \sin x$.

29. $y'' - 4y' + 3y = x e^{-x}$.   30. $y'' - 4y = \cosh x$.

31. $y'' + 4y = \sin^2 x$.   32. $y'' + 4y = x \sin 2x$.

33. $y'' + y = \cos^2 x$.   34. $y'' + 2y' + 2y = e^{-x} \cos x$.

35. $y'' + 2y' + y = e^{-x}$.   36. $y'' + 2y' + y = x e^{-x} + 6$.

37. $y'' - y' - 2y = x^2 - 4$.   38. $y'' + y' - 2y = 2x^2 + x e^{x}$.

39. $y' + 2y = 2x + \sin x$.   40. $y' + 2y = \sin x$.

41. $2y' - y = e^{2x}$.   42. $y' - 7y = e^{x} + \cos x$.

43. $y'' + 9y = \sum_{n=1}^{6} \frac{\sin nx}{n}$.   44. $y'' + y = \sum_{n=1}^{6} \frac{\sin 2nx}{n}$.

In Exercises 45–54, find the solution of the associated homogeneous equation and state the form of a particular solution.   Do not solve for the coefficients.   Be sure to modify by x or x^2 as appropriate.

45. $y'' - 4y' + 3y = e^{2x} \sinh x$.   46. $y'' - 4y' + 3y = e^{x} \sinh 2x$.

47. $y'' + 2y' + 2y = \cos x + 6x^2 - e^{-x} \sin x$.

48. $y^{(1)} - y = \cosh x + \cosh 2x$.

49. $y'' - 3\,y' + 2\,y = 3x^{-1}e^{x} + x\,e^{-2x}\cos 3x$.

50. $y'' - 3\,y' + 2\,y = x^{4}e^{-x} + 7\,x^{2}e^{2x}$.

51. $y'' + 4\,y = e^{2x}(\sin 2x + 2\cos 2x)$.

52. $y'' + 4\,y = x\sin 2x + 2\,e^{2x}\cos 2x$.

53. $y'' - 2\,y'' + y = 6x - e^{x}$. **54.** $y'' - 3\,y' + 2\,y = e^{x} + 3\,e^{-2x}$.

In Exercises 55–60, find the general solution of the given equation. Since an arbitrary parameter appears in the equation, you have to distinguish separate cases.

55. $y'' - 4y' + 3y = e^{ax}$. **56.** $y'' - 4y' + 4y = e^{ax}$.

57. $y'' + 4y = \cos\omega x$. **58.** $y'' + 9y = \sin\omega x$.

59. $y'' + \omega^{2}y = \sin 2x$. **60.** $y'' + by' + y = \sin x$.

In Exercises 61–70, solve the given initial value problem.

61. $y'' - 4\,y = 0$, $y(0) = 0$, $y'(0) = 3$.

62. $y'' + 2\,y' + y = 0$, $y(0) = 2$, $y'(0) = -1$.

63. $4y'' - 4\,y' + y = 0$, $y(0) = -1$, $y'(0) = 1$.

64. $y'' + y = 0$, $y(\pi) = 1$, $y'(\pi) = 0$.

65. $y'' - 5\,y' + 6\,y = e^{x}$, $y(0) = 0$, $y'(0) = 0$.

66. $2\,y'' - 3\,y' + y = \sin x$, $y(0) = 0$, $y'(0) = 0$.

67. $y'' - 4\,y' + 3\,y = x\,e^{-x}$, $y(0) = 0$, $y'(0) = 1$.

68. $y'' - 4\,y = \cosh x$, $y(0) = 1$, $y'(0) = 0$.

69. $y'' + 4y = \cos 2x$, $y(\pi/2) = 1$, $y'(\pi/2) = 0$.

70. $y'' + 9\,y = \sum_{n=1}^{6}\frac{\sin nx}{n}$, $y(0) = 0$, $y'(0) = 2$.

Integrals via undetermined coefficients. *In Exercises 71–74, compute $\int g(x)\,dx$ by solving $y' = g(x)$ using the method of undetermined coefficients.*

71. $g(x) = e^{x}\sin x$. **72.** $e^{x}\cos 2x$.

73. $g(x) = e^{ax}\cos bx$. **74.** $e^{ax}\sin bx$.

Given n numbers $\lambda_{1}, \lambda_{2}, \ldots, \lambda_{n}$, we define the **Vandermonde determinant** $V(\lambda_{1}, \lambda_{2}, \ldots, \lambda_{n})$ to be

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \lambda_{1} & \lambda_{2} & \cdots & \lambda_{n} \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_{1}^{n-1} & \lambda_{2}^{n-1} & \cdots & \lambda_{n}^{n-1} \end{vmatrix}.$$

75. Compute $V(\lambda_{1}, \lambda_{2})$ and show that it is nonzero if and only if $\lambda_{1} \neq \lambda_{2}$.

76. Show that $V(\lambda_{1}, \lambda_{2}, \lambda_{3}) = (\lambda_{3} - \lambda_{2})(\lambda_{3} - \lambda_{1})(\lambda_{2} - \lambda_{1})$ and conclude that $V(\lambda_{1}, \lambda_{2}, \lambda_{3}) \neq 0$ if and only if all the λ's are distinct.

77. It is a fact that $V(\lambda_{1}, \lambda_{2}, \ldots, \lambda_{n}) \neq 0$ if and only if all the λ's are distinct (see Exercise 78). Use this fact to prove that the Wronskian $W(e^{\lambda_{1}x}, e^{\lambda_{2}x}, \ldots, e^{\lambda_{n}x}) \neq 0$ if all the λ's are distinct.

78. (a) Based on Exercise 76, guess the formula for $V(\lambda_1, \lambda_2, \ldots, \lambda_n)$. Verify your guess for $n = 2, 3, 4$.
(b) Using (a), show that $V(\lambda_1, \lambda_2, \ldots, \lambda_n) \neq 0$ if and only if all the λ's are distinct.

A.3   Linear Ordinary Differential Equations with Nonconstant Coefficients

In the previous two sections we saw that the general solution of the second-order linear differential equation

$$(1) \qquad\qquad y'' + p(x)y' + q(x)y = g(x)$$

is of the form $y = y_h + y_p$, where y_h is the general solution of the associated homogeneous equation and y_p is any particular solution of (1). We obtained a complete description of the solution when the coefficient functions are constants and g is of a special form. In this section we present two general methods for handling the cases not covered by the previous section. The first one is usually applied when solving for y_h, and the second one can be used to find y_p. We end the section by applying these methods to solve the important class of Euler equations.

Reduction of Order

For second-order linear differential equations, we know that

$$y_h = c_1 y_1 + c_2 y_2,$$

where y_1 and y_2 are linearly independent solutions of the homogeneous equation

Note that the equation is in standard form. That is, the leading coefficient is 1.

$$(2) \qquad\qquad y'' + p(x)y' + q(x)y = 0.$$

Suppose that we know one nontrivial solution to (2), say y_1. The method of reduction of order allows us to find a second solution y_2 such that y_1 and y_2 are independent.

REDUCTION OF ORDER FORMULA

A second linearly independent solution of (2) given y_1 is

$$(3) \qquad\qquad y_2 = y_1 \int \frac{e^{-\int p(x)\,dx}}{y_1^2}\,dx.$$

Before you apply this formula, be sure that your equation is in standard form.

Since we are seeking only one independent solution, we can assign any fixed values to the constants of integration appearing in this formula. We will often neglect these constants.

Proof We know that cy_1 is a solution of (2). But of course this solution is linearly dependent with y_1. The idea is to find a nonconstant function $v(x)$ such that

$v(x)y_1$ is a solution. Substituting $y = v(x)y_1$ into (2), we get

$$
\begin{aligned}
y'' + p(x)y' + q(x)y &= (y_1 v'' + 2y_1' v' + y_1'' v) + p(x)(y_1 v' + y_1' v) + q(x)y_1 v \\
&= y_1 v'' + (2y_1' + p(x)y_1)v' + (y_1'' + p(x)y_1' + q(x)y_1)v \\
&= y_1 v'' + (2y_1' + p(x)y_1)v'
\end{aligned}
$$

because $y_1'' + p(x)y_1' + q(x)y_1 = 0$. Hence, for $y = vy_1$ to be a solution to (2), v must satisfy $y_1 v'' + (2y_1' + p(x)y_1)v' = 0$. This is a first-order equation in v' (hence the name *reduction of order*). Equivalently,

$$
v'' + \left(2\frac{y_1'}{y_1} + p(x)\right)v' = 0,
$$

and thus by Theorem 1, Section A.1, we find

$$
v'(x) = \frac{e^{-\int p(x)\, dx}}{y_1^2}.
$$

(We have taken the constant in this theorem to be 1, since we are only interested in one solution.) Hence (3) follows by integrating and multiplying by y_1. The linear independence of y_1 and y_2 can be checked by computing the Wronskian. See Exercise 41.   ∎

EXAMPLE 1   Reduction of order in the presence of a double root

Given that $y_1 = e^x$ is a solution to $y'' - 2y' + y = 0$, a second linearly independent solution is obtained from (3), as follows (Figure 1):

$$
y_2 = e^x \int \frac{e^{2x}}{(e^x)^2}\, dx = xe^x.
$$
∎

The characteristic equation in Example 1 has 1 as a double root. We could have used the methods of the previous section to write down the second solution. However, the point of Example 1 is to show how reduction of order can be used to derive such solutions.

Figure 1 Solutions in Example 1.

EXAMPLE 2   Applying the reduction of order formula

Given that e^x is a solution of $xy'' - (1+x)y' + y = 0$, a second linearly independent solution is obtained by appealing to (3). We have

$$
y_2 = e^x \int \frac{e^{\int(\frac{1}{x}+1)\, dx}}{(e^x)^2}\, dx = e^x \int \frac{xe^x}{(e^x)^2}\, dx = e^x \int xe^{-x}\, dx.
$$

Observe that before evaluating (3), we must put the equation in standard form. Integrating by parts, we find

$$
y_2 = e^x(-xe^{-x} - e^{-x}) = -x - 1.
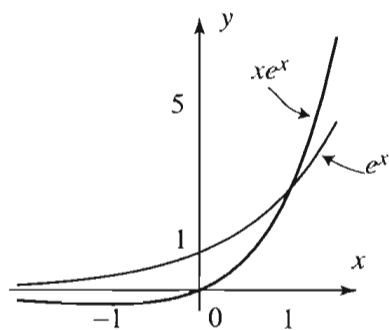$$

At this point, we may use $y_2 = x + 1$.   ∎

$$
\int xe^{-x}\, dx
$$

$$
= -xe^{-x} - e^{-x} + C.
$$

Variation of Parameters

In this part, we suppose that we know the solution of the associated homogeneous equation (2) and describe a general method for generating a particular solution of (1). This method is called **variation of parameters**.

VARIATION OF PARAMETERS FORMULA

A particular solution of (1) is given by

$$(4) \qquad y_p = y_1 \int \frac{-y_2 g(x)}{W(y_1, y_2)} \, dx + y_2 \int \frac{y_1 g(x)}{W(y_1, y_2)} \, dx,$$

where y_1 and y_2 are linearly independent solutions of (2), and $W(y_1, y_2) = y_1 y_2' - y_1' y_2$ is the Wronskian of y_1 and y_2.

As in the reduction of order method, we can neglect the constants of integration.

Proof   Since $c_1 y_1 + c_2 y_2$ is a solution of the homogeneous equation, our hope is that by allowing functions instead of constants we can generate a solution of the nonhomogeneous equation. We thus try

$$(5) \qquad y = u_1(x) y_1 + u_2(x) y_2$$

and solve for u_1 and u_2. Since we have two unknown functions and only one equation to satisfy, we are free to impose one additional relation between u_1 and u_2. As you will see, the following condition simplifies the computation:

$$(6) \qquad u_1'(x) y_1 + u_2'(x) y_2 = 0.$$

We now compute using (5) and (6):

$$y' = u_1(x) y_1' + u_2(x) y_2'; \quad y'' = u_1(x) y_1'' + u_2(x) y_2'' + u_1'(x) y_1' + u_2'(x) y_2'.$$

Substituting these and (5) in the left side of (1) we get

$$
\begin{aligned}
y'' + p(x) y' + q(x) y &= (u_1(x) y_1'' + u_2(x) y_2'' + u_1'(x) y_1' + u_2'(x) y_2') \\
&\quad + p(x)(u_1(x) y_1' + u_2(x) y_2') + q(x)(u_1(x) y_1 + u_2(x) y_2) \\
&= u_1(x) \underbrace{(y_1'' + p(x) y_1' + q(x) y_1)}_{=0} \\
&\quad + u_2(x) \underbrace{(y_2'' + p(x) y_2' + q(x) y_2)}_{=0} + u_1'(x) y_1' + u_2'(x) y_2'.
\end{aligned}
$$

Therefore, recalling (6), we will have a solution if u_1 and u_2 satisfy

$$
\begin{cases}
y_1 u_1'(x) + y_2 u_2'(x) = 0, \\
y_1' u_1'(x) + y_2' u_2'(x) = g(x).
\end{cases}
$$

The determinant of this system is precisely $W(y_1, y_2)$, which is nonzero since y_1 and y_2 are linearly independent. Solving this system, we get the unique solutions

$$u_1' = \frac{-y_2 g(x)}{W(y_1, y_2)} \quad \text{and} \quad u_2' = \frac{y_1 g(x)}{W(y_1, y_2)}.$$

Integrating and substituting in (5) yields (4).   ■

EXAMPLE 3   Variation of parameters
Given that $y_1 = x$ and $y_2 = x^4$ are solutions of $x^2 y'' - 4xy' + 4y = 0$, find a particular solution of

$$x^2 y'' - 4xy' + 4y = 3x^2.$$

Solution We have

$$W(y_1, y_2) = W(x, x^4) = 3x^4.$$

We put the equation in standard form, apply (4), and get

$$y_p = x \int \frac{-3x^4}{3x^4} \, dx + x^4 \int \frac{3x}{3x^4} \, dx = -x^2 - \frac{1}{2} x^2 = -\frac{3}{2} x^2. \qquad ■$$

We end this section with a discussion of a class of differential equations with important applications.

Euler Equations

The differential equation

$$(7) \qquad\qquad x^2 y'' + \alpha \, xy' + \beta y = 0,$$

where α and β are constants, is known as **Euler's equation**. It is the simplest example of a second-order linear differential equation with nonconstant coefficients for which we have an explicit solution. Motivated by the first-order version of this equation, $xy' + \alpha y = 0$, which has solution $y = x^{-\alpha}$, we try

$$(8) \qquad\qquad y = x^r$$

as a solution of (7). Plugging x^r into the equation, it follows that r must be a root of

$$(9) \qquad\qquad r(r-1) + \alpha r + \beta = 0.$$

This quadratic equation, known as the **indicial equation**, is the key to solving (7), in the same way that the characteristic equation is the key to solving an equation with constant coefficients. As expected, the solutions will depend on the nature of the roots, referred to as the **indicial roots**.

If we put the general Euler's equation in standard form, the coefficients of y' and y are not defined at $x = 0$. Because of this problem at 0, the cases $x > 0$ and $x < 0$ are to be treated separately. In fact, in most applications we are only interested in the case $x > 0$. For completeness, in the following box we present the solution of Euler's equation in both cases using the absolute value.

**GENERAL
SOLUTION OF
EULER'S EQUATION**

Consider Euler's equation (7) with indicial equation (9), which we write as

$$(10) \qquad\qquad r^2 + (\alpha - 1)r + \beta = 0.$$

Let r_1 and r_2 denote the indicial roots. The general solution y of this equation is given by the following cases.

Case I If r_1 and r_2 are distinct real roots, then

$$y = c_1|x|^{r_1} + c_2|x|^{r_2}.$$

Case II If $r_1 = r_2$, then

$$y = (c_1 + c_2 \ln|x|)|x|^{r_1}.$$

Case III If r_1 and r_2 are complex conjugate roots with $r_1 = a + ib$, then

$$y = |x|^a[c_1 \cos(b \ln|x|) + c_2 \sin(b \ln|x|)].$$

Clearly when $x > 0$ we may drop the absolute values.

Proof For clarity's sake, we take $x > 0$. Case I follows immediately from (8) and the fact that $W(x^{r_1}, x^{r_2}) = (r_2 - r_1)x^{r_1+r_2-1} \neq 0$ for $x > 0$. Case II is derived via reduction of order. Using $y_1 = x^{r_1}$ in (3), we get

$$y_2 = x^{r_1} \int \frac{e^{-\int \frac{\alpha}{x} dx}}{x^{2r_1}} dx = x^{r_1} \int x^{-\alpha - 2r_1} dx.$$

But because r_1 is a double root of the indicial equation (10), we have $2r_1 = r_1 + r_2 = -(\alpha - 1)$. So $x^{-\alpha-2r_1} = x^{-1}$, and the integral evaluates to $\ln x$, implying $y = x^{r_1} \ln x$.

In Case III, two linearly independent complex-valued solutions are formally given by x^{a+ib} and x^{a-ib}. We interpret these as $e^{\ln x(a+ib)}$ and $e^{\ln x(a-ib)}$ and proceed to derive real-valued solutions. From Euler's identity, we have

$$e^{\ln x(a+ib)} = e^{a \ln x} e^{ib \ln x} = x^a[\cos(b \ln x) + i \sin(b \ln x)],$$

and, similarly,

$$e^{\ln x(a-ib)} = x^a[\cos(b \ln x) - i \sin(b \ln x)].$$

Taking linear combinations, we arrive at the desired real solutions as we did previously when dealing with constant coefficient equations having complex characteristic roots. Linear independence follows by computing the Wronskian and is left as Exercise 42. ∎

There is a close similarity between the solution to Euler's equation and the solution to the constant coefficient equation. Indeed, the change of variables $t = \ln x$ in Euler's equation transforms it to an equation with
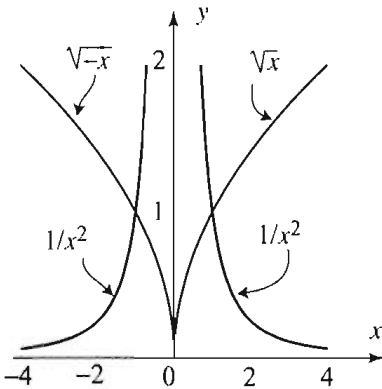
constant coefficients. This provides an alternative derivation of the general solution of Euler's equation. See Exercise 43.

EXAMPLE 4   Euler's equation

Solve $2x^2y'' + 5xy' - 2y = 0$ for $x > 0$.

Solution We rewrite the equation as $x^2y'' + \frac{5}{2}xy' - y = 0$. From (10), the indicial equation is $r^2 + \frac{3}{2}r - 1 = 0$, which factors as $(r - \frac{1}{2})(r + 2) = 0$. We get the indicial roots $r_1 = \frac{1}{2}$, $r_2 = -2$. Thus by Case I, the general solution is $y = c_1\sqrt{x} + \frac{c_2}{x^2}$. The general solution is illustrated in Figure 2. ∎

Note that when $x < 0$, the solution in Example 4 becomes

$$y = c_1\sqrt{-x} + \frac{c_2}{x^2}.$$

Figure 2 Solutions in Example 4. Notice that $\frac{1}{x^2}$ is not defined at $x = 0$ and $\sqrt{x}$ is not differentiable at $x = 0$. These solutions are valid for $x > 0$, where the differential equation is defined.

Exercises A.3

In Exercises 1-20, verify that the given function is a solution of the given equation, and then find the general solution using the reduction of order formula.

1. $y'' + 2y' - 3y = 0$, $y_1 = e^x$.

2. $y'' - 5y' + 6y = 0$, $y_1 = e^{3x}$.

3. $xy'' - (3+x)y' + 3y = 0$, $y_1 = e^x$.

4. $xy'' - (2-x)y' - 2y = 0$, $y_1 = e^{-x}$.

5. $y'' + 4y = 0$, $y_1 = \cos 2x$.

6. $y'' + 9y = 0$, $y_1 = \sin 3x$.

7. $y'' - y = 0$, $y_1 = \cosh x$.

8. $y'' + 2y' + y = 0$, $y_1 = e^{-x}$.

9. $(1 - x^2)y'' - 2xy' + 2y = 0$, $y_1 = x$.

10. $(1 - x^2)y'' - 2xy' = 0$, $y_1 = 1$.

11. $x^2y'' + xy' - y = 0$, $y_1 = x$.

12. $x^2y'' - xy' + y = 0$, $y_1 = x$.

13. $x^2y'' + xy' + y = 0$, $y_1 = \cos(\ln x)$.

14. $x^2y'' + 2xy' + \frac{1}{4}y = 0$, $y_1 = 1/\sqrt{x}$.

15. $xy'' + 2y' + 4xy = 0$, $y_1 = \frac{\sin 2x}{x}$.

16. $xy'' + 2y' - xy = 0$, $y_1 = \frac{e^x}{x}$.

17. $xy'' + 2(1-x)y' + (x-2)y = 0$, $y_1 = e^x$.

18. $(x-1)^2y'' - 3(x-1)y' + 4y = 0$, $y_1 = (x-1)^2$.

19. $x^2y'' - 2xy' + 2y = 0$, $y_1 = x^2$.

20. $(x^2 - 2x)y'' - (x^2 - 2)y' + 2(x-1)y = 0$, $y_1 = e^x$.

In Exercises 21–30, find the general solution of the given equation using the method of variation of parameters.   Take $x > 0$.

21. $y'' - 4y' + 3y = e^{-x}$.

22. $y'' - 15y' + 56y = e^{7x} + 12x$.

23. $3y'' + 13y' + 10y = \sin x$.

24. $y'' + 3y = x$.

25. $y'' + y = \sec x$.

26. $y'' + y = \sin x + \cos x$.

27. $xy'' - (1 + x)y' + y = x^3$.
[Hint: $y_1 = 1 + x, y_2 = e^x$.]

28. $xy'' - (1 + x)y' + y = x^4 e^x$.
[Hint: Exercise 27.]

29. $x^2 y'' + 3xy' + y = \sqrt{x}$.

30. $x^2 y'' + xy' + y = x$.

In Exercises 31–40, solve the given Euler equation.   Take $x > 0$, unless otherwise stated.

31. $x^2 y'' + 4xy' + 2y = 0$.

32. $x^2 y'' + xy' - 4y = 0$.

33. $x^2 y'' + 3xy' + y = 0$.

34. $4x^2 y'' + 8xy' + y = 0$.

35. $x^2 y'' + xy' + 4y = 0$.

36. $4x^2 y'' + 4xy' + y = 0$.

37. $x^2 y'' + 7xy' + 13y = 0$.

38. $x^2 y'' - xy' + 5y = 0$.

39. $(x - 2)^2 y'' + 3(x - 2)y' + y = 0$
$(x > 2)$.
[Hint: Let $t = x - 2$.]

40. $(x + 1)^2 y'' + (x + 1)y' + y = 0$
$(x > -1)$.
[Hint: Let $t = x + 1$.]

41. Compute $W(y_1, y_2)$ with y_2 given by (3) and conclude that the reduction of order formula yields a second linearly independent solution.

42. Let y_1 and y_2 be the solutions of Euler's equation in Case III. Show that $W(y_1, y_2) = x^{2a-1}$ and conclude that y_1 and y_2 are linearly independent for $x > 0$.

43 Show that the change of variables $t = \ln x$ $(x > 0)$ transforms Euler's equation (7) to the equation

$$\frac{d^2 y}{dt^2} + (\alpha - 1)\frac{dy}{dt} + \beta y = 0$$

with constant coefficients.

44. An alternative solution of Euler's equation. Using Exercise 43 and the solution of (6), Section A.2, derive the three cases of the general solution of Euler's equation.

45.   Reduction of order formula from Abel's formula.
(a) Use Abel's formula (Theorem 2, Section A.1) to conclude that

$$y_1 y_2' - y_1' y_2 = Ce^{-\int p(x)\,dx},$$

where y_1 and y_2 are any two solution of (2).
(b) Given y_1, set $C = 1$ in (a) and solve the resulting first-order differential equation in y_2, thereby deriving (3).

46. Reduction of order for nonhomogeneous equations. In this exercise we demonstrate that the method of reduction of order also applies to nonhomogeneous equations given a solution y_1 to the associated homogeneous equation. Thus, given y_1, we may solve (1) directly without recourse to the method of variation of parameters.

(a) Show that if we want to solve (1) and carry out the proof of (3) we arrive at the equation

$$v'' + \left(\frac{y_1'}{y_1} + p(x)\right) v' = \frac{g(x)}{y_1}.$$

(b) Solve the equation using **Theorem 1, Section A.1**, and conclude that the general solution of (1) is

$$y = c_1 y_1 + c_2\, y_1 \int \frac{e^{-\int p(x)\, dx}}{y_1^2}\, dx$$

$$+ y_1 \int \frac{e^{-\int p(x)\, dx}}{y_1^2}\, dx \left(\int y_1 e^{\int p(x)\, dx} g(x)\, dx\right)\, dx$$

where the last two occurrences of $\int p(x)\, dx$ represent the same antiderivative of $p(x)$.

In Exercises 47–50, find the general solution of the given equation by using the method of Exercise 46. To get a good feel for Exercise 46, we suggest that you repeat its proof with at least one of the following exercises.

47. $y'' - 4y' + 3y = e^x$, $y_1 = e^x$.

48. $x^2 y'' + 3xy' + y = \sqrt{x}$, $y_1 = \frac{1}{x}$.

49. $3\, y'' + 13\, y' + 10\, y = \sin x$, $y_1 = e^{-x}$.

50. $x\, y'' - (1 + x)y' + y = x^3$, $y_1 = e^x$.

A.4   The Power Series Method, Part I

Power series will be used to solve ordinary differential equations, including many important differential equations with nonconstant coefficients, such as the Legendre and Bessel equations. In this section we review the basic properties of power series, and present some techniques that will be needed in the next sections.

A **power series** is a series of the form

$$(1) \qquad \sum_{m=0}^{\infty} a_m (x - a)^m = a_0 + a_1(x - a) + a_2(x - a)^2 + \cdots,$$

where $a, a_0, a_1, a_2, \ldots$ are constants and x is a variable. The series is said to be **centered** at a. If the series is centered at $a = 0$, it takes on the simpler form

$$(2) \qquad \sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + \cdots.$$

Given the power series (1), we define a function $f(x)$ whose domain is the set of all x for which the series converges, and whose value at every such x is the sum of the series for that x. We say that the power series represents the function $f(x)$.

Recall from calculus that certain functions have a Taylor series representation centered at $x = a$ of the form

$$(3) \quad f(x) = f(0) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(k)}(a)}{k!}(x-a) + \cdots .$$

Thus a Taylor series is an example of a power series where the coefficients are uniquely determined by $f(x)$ and its derivatives. In fact, we have $a_0 = f(a)$ and $a_k = f^{(k)}(a)/k!$. The following list gives Taylor series representations of some elementary functions, together with the domains for which the series expansions are valid. All the series are centered at 0.

$$\frac{1}{1-x} = \sum_{m=0}^{\infty} x^m = 1 + x + x^2 + x^3 + \cdots , \qquad |x| < 1;$$

$$e^x = \sum_{m=0}^{\infty} \frac{x^m}{m!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots , \qquad -\infty < x < \infty;$$

$$\sin x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots , \qquad -\infty < x < \infty;$$

$$\cos x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots , \qquad -\infty < x < \infty;$$

$$\ln(1+x) = \sum_{m=1}^{\infty} \frac{(-1)^{m+1} x^m}{m} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots , \qquad |x| < 1;$$

$$\tan^{-1} x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{2m+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots , \qquad |x| < 1;$$

$$\sinh x = \sum_{m=0}^{\infty} \frac{x^{2m+1}}{(2m+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots , \qquad -\infty < x < \infty;$$

$$\cosh x = \sum_{m=0}^{\infty} \frac{x^{2m}}{(2m)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots , \qquad -\infty < x < \infty.$$

Series converges for $|x - a| < R$.
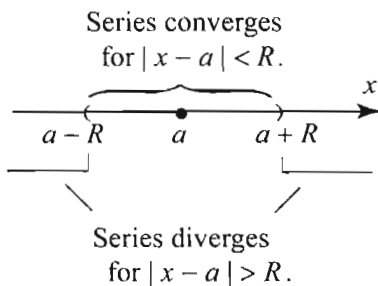
Series diverges for $|x - a| > R$.

Figure 1 The interval of convergence of a power series centered at the point $x = a$. The series may converge or diverge at the endpoints $x = a \pm R$.

As the preceding list indicates, a power series need not converge for all x. We recall the following facts about the convergence of power series.

- The power series (1) always converges at its center $x = a$.

- If the series converges for values of x other than a, then these values form an interval centered at a, called the **interval of convergence**. If the interval is finite, then there is a number $R > 0$, called the **radius of convergence**, such that the series converges at all x for which $|x - a| < R$, and diverges at all x for which $|x - a| > R$ (Figure 1). If the series converges only at $x = a$, we set $R = 0$, and if the series converges for all x, we set $R = \infty$.

A function defined by a power series is said to be **analytic** on the interval of convergence. (In Section 12.5 we used the term *analytic* to describe a

complex function that is differentiable in an open subset of the plane. A basic result from complex analysis tells us that this property is equivalent to having a power series expansion. So the usage of the term *analytic* in this section is consistent with our previous usage in Section 12.5.)

A useful test for finding the radius of convergence of a power series is the ratio test, as illustrated by the following examples. (In some cases the root test may be more convenient.)

EXAMPLE 1   Radius and interval of convergence
Consider the power series

$$\sum_{m=1}^{\infty} \frac{(-1)^m m}{4^m} x^{2m} = -\frac{1}{4}x^2 + \frac{1}{8}x^4 - \frac{3}{64}x^6 + \cdots .$$

Using the ratio test, we have that the series converges whenever the limit

$$\lim_{m \to \infty} \left| \frac{(m+1)}{4^{m+1}} x^{2(m+1)} \Big/ \frac{m}{4^m} x^{2m} \right| = \lim_{m \to \infty} \left(\frac{m+1}{4m} \right) x^2 = \frac{x^2}{4}$$

is less than 1. That is, $x^2/4 < 1$ or $x^2 < 4$. Thus, the interval of convergence is $|x| < 2$, or $(-2, 2)$. Since the series is centered at 0, the radius of convergence is 2. The behavior of the series at the endpoints, $x = \pm 2$, is not determined by the ratio test. In this example, you can check that the series diverges at $x = \pm 2$. Thus we can define a function for all x in $(-2, 2)$ by the power series, as follows: $f(x) = \sum_{m=1}^{\infty} \frac{(-1)^m m}{4^m} x^{2m}$. For each value of x in $(-2, 2)$, the series converges to a real number, which is $f(x)$.   ∎

EXAMPLE 2   A power series that converges for all x
Find the radius of convergence of the series

$$\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{2^m (m!)^2} x^m = \frac{x}{2} - \frac{x^2}{16} + \frac{x^3}{288} - \cdots .$$

Solution Evaluating the limit obtained from the ratio test, we find

$$\lim_{m \to \infty} \left| \frac{x^{m+1}}{2^{m+1}[(m+1)!]^2} \Big/ \frac{x^m}{2^m (m!)^2} \right| = \lim_{m \to \infty} \left(\frac{|x|}{2(m+1)^2} \right) = 0$$

for all x. Thus the series converges for all x. The radius of convergence is ∞, and the interval of convergence is $(-\infty, \infty)$.   ∎

EXAMPLE 3   A power series that converges only at its center
For the power series

$$\sum_{m=1}^{\infty} m^m (x-1)^m = (x-1) + 4(x-1)^2 + 27(x-1)^3 + 256(x-1)^4 + \cdots ,$$

we use the root test and get

$$\lim_{m \to \infty} \sqrt[m]{|m^m (x-1)^m|} = \lim_{m \to \infty} m|x-1| = \infty$$

for all $x \neq 1$. Thus the series diverges for all $x \neq 1$. The series is clearly convergent when $x = 1$. The radius of convergence is 0.   ∎

Operations on Power Series

Power series are like polynomials of "infinite degree," and like polynomials they can be added, multiplied, differentiated, and integrated term-by-term. We now recall how these operations apply.

Linear Combinations and Products. Consider the power series representations

$$f(x) = \sum_{m=0}^{\infty} a_m (x-a)^m \quad \text{and} \quad g(x) = \sum_{m=0}^{\infty} b_m (x-a)^m,$$

with common center at a and radii of convergence $R_1 > 0$ and $R_2 > 0$, respectively. Then it is not difficult to show that for any real numbers α and β, we have

$$\alpha f(x) + \beta g(x) = \sum_{m=0}^{\infty} (\alpha\, a_m + \beta\, b_m)(x-a)^m.$$

The multiplication of power series is done in much the same way as for polynomials:

$$
\begin{aligned}
f(x) \cdot g(x) \\
&= \left(a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots\right) \cdot \left(b_0 + b_1(x-a) + b_2(x-a)^2 + \cdots\right) \\
&= a_0 + b_0 + \left(a_0 b_1 + a_1 b_0\right)(x-a) + \left(a_0 b_2 + a_1 b_1 + a_2 b_0\right)(x-a)^2 + \cdots \\
&= \sum_{m=0}^{\infty} \left(a_0 b_m + a_1 b_{m-1} + \cdots + a_m b_0\right)(x-a)^m.
\end{aligned}
$$

The power series of the linear combination $\alpha f + \beta g$ and the product $f \cdot g$ have radii of convergence at least as large as the smaller of R_1 and R_2.

Differentiation and Integration of Power Series. A power series

$$(4) \qquad f(x) = \sum_{m=0}^{\infty} a_m (x-a)^m \quad (|x-a| < R, \ R > 0)$$

can be differentiated term-by-term to yield

$$(5) \qquad f'(x) = \sum_{m=1}^{\infty} m a_m (x-a)^{m-1} \quad (|x-a| < R).$$

Moreover, the series (4) and (5) have the same interval of convergence. The proof is sketched in the exercises; it uses the notion of uniform convergence and results about series of functions from Section 2.9. Since the differentiated series (5) is itself a power series with interval of convergence equal to the interval of convergence of the series (4), we can differentiate it again, and again, and conclude that f is infinitely differentiable.

If we differentiate the power series k times, we find

$$f^{(k)}(x) = \sum_{m=k}^{\infty} m(m-1)\cdots(m-k+1)a_m(x-a)^{m-k} \quad (|x-a| < R).$$

Setting $x = a$, all the terms cancel, except for $m = k$, and we get

$$f^{(k)}(a) = \overbrace{k(k-1)\cdots(k-k+1)}^{=k!}a_k \quad \Rightarrow \quad a_k = \frac{f^{(k)}(a)}{k!}.$$

Substituting back into (4), we obtain the **Taylor series expansion** of $f(x)$:

$$(6) \qquad f(x) = \sum_{m=1}^{\infty} \frac{f^{(k)}(a)}{m!}(x-a)^m \quad (|x-a| < R).$$

This shows that every power series with a positive radius of convergence is the Taylor series of the function $f(x)$ that it represents. (In general, you may not be able to express $f(x)$ in terms of familiar elementary functions, as shown in the list at the outset of the section.)

The power series (4) can be integrated term-by-term on any closed and bounded subintervals of the interval of convergence, as follows. Let $[b, c]$ be a closed interval in the interval of convergence; then

$$\int_b^c f(t)\, dt \;=\; \sum_{m=0}^{\infty} a_m \int_b^c (t-a)^m$$

$$=\; \sum_{m=0}^{\infty} \frac{a_m}{m+1}\big[(c-a)^{m+1} - (b-a)^{m+1}\big].$$

In particular, if x is in the interval of convergence, then

$$\int_a^x f(t)\, dt = \sum_{m=0}^{\infty} \frac{a_m}{m+1}(x-a)^{m+1},$$

because

$$\int_a^x (t-a)^m\, dt = \frac{1}{m+1}(t-a)^{m+1}\Big|_a^x = \frac{1}{m+1}(x-a)^{m+1}.$$

The proof is also sketched in the exercises. Let us illustrate the usefulness of term-by-term differentiation and integration by deriving new power series from known ones, without performing excessive computations.

EXAMPLE 4   Differentiation and integration term-by-term
(a) Start with the geometric series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{m=0}^{\infty} x^m \quad (-1 < x < 1).$$

Differentiate both sides of the the first equality:

$$\frac{1}{(1-x)^2} = 0 + 1 + 2x + 3x^2 + \cdots = \sum_{m=1}^{\infty} mx^{m-1} \quad (-1 < x < 1).$$

This gives the power series expansion of the function $1/(1-x)^2$. Observe the following two facts:

- The differentiated series has the same radius of convergence as the original series.

- The index in the differentiated series starts at $m = 1$, whereas in the original series it starts at $m = 0$. This is clear since the term corresponding to $m = 0$ in the differentiated series is 0.

(b) Start with the geometric series in (a). For x in $(-1, 1)$, we have

$$\int_0^x \frac{1}{1-t} dt = \int_0^x \left(1 + t + t^2 + t^3 + \cdots\right) dt;$$

$$-\ln(1-t)\Big|_0^x = \left(t + \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{4} + \cdots\right)\Big|_0^x$$

$$-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots.$$

This gives the power series representation of $\ln(1-x)$, which is valid in the interval $-1 < x < 1$. Note that we could have obtained this representation by replacing x by $-x$ in the expansion of $\ln(1+x)$ from the list given at the beginning of the section.   ∎

In the preceding examples, we used various operations on known series to derive new power series expansions. Thus we claimed to have found the power series expansions of $1/(1-x)^2$ and $\ln(1-x)$. The question that comes to mind is whether there are other power series representations of these functions. The answer is provided by the following important result.

The Identity Principle. Suppose $f(x) = \sum_{m=0}^{\infty} a_m(x-a)^m$ and $g(x) = \sum_{m=0}^{\infty} b_m(x-a)^m$, where both series have positive radii of convergence. If $f(x) = g(x)$ for all x in an interval containing a, then $a_m = b_m$ for all m. In other words, a power series representation is unique.

This is an immediate consequence of the fact that the coefficients in the power series are determined from the values of the function and its derivatives at $x = a$ (Taylor's formula). So if $f = g$ in an interval containing a, then $f^{(m)}(a) = g^{(m)}(a)$ for all m, which implies that the coefficients are the same in both power series.

We will often use the identity principle to determine the coefficients of a power series that appears in an equation, as the following examples illustrate.

EXAMPLE 5   The identity principle

Determine a_m if for all x in the interval $(-1, 1)$, we have

$$\sum_{m=0}^{\infty}(a_m - 2m)x^m = 0.$$

Solution By the identity principle, there is only one power series that vanishes identically on $(-1, 1)$, and this is the power series whose coefficients are all equal to 0. Thus $a_m - 2m = 0$ for all m, and so $a_m = 2m$. ∎

EXAMPLE 6   Equation involving a power series

Determine a_m if for all x in the interval $(-1, 1)$, we have

$$\sum_{m=0}^{\infty} a_m x^m + \sum_{m=0}^{\infty} m a_m x^m = \frac{1}{1-x}.$$

Solution Combining the two series on the left, the equation becomes

$$\sum_{m=0}^{\infty}(1+m)a_m x^m = \frac{1}{1-x}.$$

To be able to proceed from this point, we express $1/(1-x)$ in a power series about 0 and use the identity principle to equate the corresponding coefficients of x^m. Thus

$$\sum_{m=0}^{\infty}(1+m)a_m x^m = \sum_{m=0}^{\infty} x^m \quad \Rightarrow \quad (1+m)a_m = 1 \text{ for all } m,$$

because all the coefficients in the expansion of $1/(1-x)$ are equal to 1. Solving the last equation, we find $a_m = 1/(1+m)$. Thus $a_0 = 1$, $a_1 = 1/2$, $a_2 = 1/3$, and so on. ∎

Composition of Power Series. We know from calculus that if $f(x)$ is differentiable at $x = a$ and $g(x)$ is differentiable at $f(a)$, then $g(f(x))$ is differentiable at a. This fact remains valid if we replace *differentiable* by *analytic*; that is, if $f(x)$ has a power series expansion centered at a and $g(x)$ has a power series expansion centered at $f(a)$, then $g(f(x))$ has a power series expansion centered at a.

As an illustration, suppose that $f(x) = (x - a)^m$ and $g(x)$ is analytic at 0 $(= f(a))$. Then the power series of $g(f(x))$ about a is obtained by substituting $(x - a)^m$ in the power series expansion of $g(x)$ about 0. For example, substituting x by x^2 in the Taylor series expansion of e^x, we obtain the Taylor series of e^{x^2},

$$e^{x^2} = \sum_{m=0}^{\infty} \frac{x^{2m}}{m!} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots \quad (-\infty < x < \infty).$$

Quotients of Analytic Functions. Suppose that f and g are analytic at a and $g(a) \neq 0$. Then the quotient f/g is also analytic at a.

To see this, we view the function $1/g$ as the composition of g with the function $1/x$. The latter being analytic everywhere **except at $x = 0$** (Exercise 25), it follows that $1/g$ is analytic at a if $g(a) \neq 0$. Now $f/g = f \cdot (1/g)$, and the desired conclusion follows since the product of two analytic functions is again analytic.

The following examples illustrate some useful techniques that can be used when computing a power series of a rational function.

EXAMPLE 7   Power series of rational functions

Find the power series expansion at $a = 0$ of the given function, and determine the radius of convergence.

(a) $\dfrac{1}{3 + 2x}$.

(b) $\dfrac{4x^2 + 8x - 4}{3 + 2x}$.

Solution (a) To find the power expansion, we try to relate the function to the geometric series, which we recall in the following form:

$$\sum_{m=0}^{\infty} u^m = \frac{1}{1 - u} \quad (|u| < 1).$$

If we can write $1/(3 + 2x)$ in the form $A/(1 - u)$ for some u, where A is a constant, then we will be done. For this purpose, we proceed as follows:

$$\frac{1}{3 + 2x} = \frac{1}{3} \frac{1}{1 - \left(-\frac{2x}{3}\right)} = \frac{1}{3} \frac{1}{1 - u},$$

where $u = -2x/3$. Thus

$$\frac{1}{3 + 2x} = \frac{1}{3} \sum_{m=0}^{\infty} \left(-\frac{2x}{3}\right)^m = \sum_{m=0}^{\infty} (-1)^m \frac{2^m x^m}{3^{m+1}}.$$

Furthermore, the series converges if and only if $|u| < 1$; equivalently, $|-2x/3| < 1$ or $|x| < 3/2$. Hence the series has radius of convergence equal to $3/2$.

(b) Dividing the numerator by the denominator, we obtain

$$\frac{4x^2 + 8x - 4}{3 + 2x} = 2x + 1 - \frac{7}{3 + 2x}.$$

Using the power series expansion from (a), we find

$$
\begin{aligned}
\frac{4x^2 + 8x - 4}{3 + 2x} &= 1 + 2x - 7 \sum_{m=0}^{\infty} (-1)^m \frac{2^m x^m}{3^{m+1}} \\
&= 1 - \frac{7}{3} + x \left(2 + \frac{14}{9}\right) - 7 \left(\frac{4}{27} x^2 - \frac{8}{243} x^3 + \cdots\right) \\
&= -\frac{4}{3} + \frac{32}{9} x - 7 \sum_{m=2}^{\infty} (-1)^m \frac{2^m x^m}{3^{m+1}}.
\end{aligned}
$$

The power series has the same radius of convergence as the one in part (a).   ■

Shifting the Index in a Power Series. When using power series to solve differential equations, we are often required to combine several power series into a single one, as we did in Example 6. Suppose you want to combine the expression

$$\sum_{m=0}^{\infty} x^{m+1} + \sum_{m=0}^{\infty} (m+1)x^m$$

into one power series. Unlike what we did in Example 6, here we cannot simply add terms with corresponding indices, because the variable x has a different power in the general term of each series. To deal with this situation, we can rewrite the first series in terms of x^m, as follows. Let $m + 1 = n$. As m varies from 0 to ∞, n varies from 1 to ∞ and the first series becomes

$$\sum_{m=0}^{\infty} x^{m+1} = \sum_{n=1}^{\infty} x^n.$$

Now rewrite the index n as m, and you have

$$\sum_{m=0}^{\infty} x^{m+1} = \sum_{m=1}^{\infty} x^m.$$

So we were able to *shift* the index of the terms without changing the power series. In general, the index m in a power series can be shifted by k units as follows:

$$\sum_{m=s}^{\infty} a_m(x-a)^{m+k} = \sum_{m=s+k}^{\infty} a_{m-k}(x-a)^m.$$

Thus, *if we shift the index of summation by changing m to $m - k$ in the general term of the series, then we must increase the starting point of the summation by k.* We also have

$$\sum_{m=s}^{\infty} a_m(x-a)^m = \sum_{m=s-k}^{\infty} a_{m+k}(x-a)^{m+k}.$$

Thus, *if we shift the index of summation by changing m to $m + k$ in the general term of the series, then we must decrease the starting point of the summation by k.*

EXAMPLE 8   Shifting indices

Write the following sum as a single power series

$$\sum_{m=1}^{\infty} mx^{m-1} + \sum_{m=0}^{\infty} 2x^{m+1}.$$

Solution Let us express both series in terms of x^m. That means we must change m to $m + 1$ in the first series, so that the exponent of x becomes $(m + 1) - 1 = m$. In the second series we must change m to $m - 1$. With this in mind, the series become

$$\sum_{m=0}^{\infty} (m + 1)x^m + \sum_{m=1}^{\infty} 2x^m.$$

We can now combine the two series as

$$(0 + 1)x^0 + \sum_{m=1}^{\infty} (m + 1)x^m + \sum_{m=1}^{\infty} 2x^m = 1 + \sum_{m=1}^{\infty} (m + 3)x^m.$$

Note how we split off the $m = 0$ term from the first series, since the second one starts at $m - 1$. ∎

The following two examples illustrate some of the basic steps in the power series method of the next section.

EXAMPLE 9   Combining power series and their derivatives
Suppose that $y = \sum_{m=0}^{\infty} a_m x^m$ is a power series with radius of convergence $R > 0$. Find a power series representation for the expression

$$(1 + x)y' - 2y.$$

Solution A power series can be differentiated term-by-term within its radius of convergence. So for $|x| < R$, we have

$$y' = \sum_{m=1}^{\infty} a_m m x^{m-1}.$$

Using that $x \sum_{m=1}^{\infty} a_m m x^{m-1} = \sum_{m=1}^{\infty} a_m m x^m$, we obtain

$$
\begin{aligned}
(1 + x)y' &= (1 + x) \sum_{m=1}^{\infty} a_m m x^{m-1} = \sum_{m=1}^{\infty} a_m m x^{m-1} + \sum_{m=1}^{\infty} a_m m x^m \\
&= \sum_{m=0}^{\infty} a_{m+1}(m + 1)x^m + \sum_{m=1}^{\infty} a_m m x^m.
\end{aligned}
$$

The second series can be started at $m = 0$, since the term corresponding to $m = 0$ is 0. So

$$
\begin{aligned}
(1 + x)y' &= \sum_{m=0}^{\infty} \left[(m + 1)a_{m+1} + m a_m \right] x^m; \\
(1 + x)y' - 2y &= \sum_{m=0}^{\infty} \left[(m + 1)a_{m+1} + m a_m \right] x^m - 2 \sum_{m=0}^{\infty} a_m x^m \\
&= \sum_{m=0}^{\infty} \left[(m + 1)a_{m+1} + (m - 2)a_m \right] x^m.
\end{aligned}
$$

∎

EXAMPLE 10   Combining power series and their derivatives
Continuing the previous example, find a power series representation of

$$x^2 y'' + (1 + x)y' - 2y.$$

Solution Differentiating the power series of y twice, we obtain

$$y'' = \sum_{m=2}^{\infty} a_m m(m-1)x^{m-2} \quad \Rightarrow \quad x^2 y'' = \sum_{m=2}^{\infty} a_m m(m-1)x^m.$$

We now appeal to the result of the previous exercise and get

$$x^2 y'' + (1 + x)y' - 2y$$
$$= \sum_{m=2}^{\infty} a_m m(m-1)x^m + \sum_{m=0}^{\infty} \left[(m+1)a_{m+1} + (m-2)a_m\right]x^m.$$

To combine the two series, we split off the terms corresponding to $m = 0$ and 1 from the second series and get

$$x^2 y'' + (x + 1)y' - 2y$$
$$= (a_1 - 2a_0) + (2a_2 - a_1) + \sum_{m=2}^{\infty}\left[a_m m(m-1) + (m+1)a_{m+1} + (m-2)a_m\right]x^m$$
$$= 2(a_2 - a_0) + \sum_{m=2}^{\infty}\left[(m+1)a_{m+1} + a_m(m^2 - 2)\right]x^m. \qquad \blacksquare$$

Exercises A.4

In Exercises 1–12, find the radius and interval of convergence of the given series.

1. $\displaystyle\sum_{m=0}^{\infty} \frac{x^m}{5m+1}.$

2. $\displaystyle\sum_{m=0}^{\infty} \frac{3^m x^m}{m!}.$

3. $\displaystyle\sum_{m=0}^{\infty} \frac{(-1)^m x^{m+2}}{2^m}.$

4. $\displaystyle\sum_{m=0}^{\infty} (x-2)^m.$

5. $\displaystyle\sum_{m=1}^{\infty} \frac{m^m x^m}{m!}.$

6. $\displaystyle\sum_{m=1}^{\infty} \frac{(m-1)(m+3)x^m}{m}.$

7. $\displaystyle\sum_{m=2}^{\infty} \frac{x^m}{\ln m}.$

8. $\displaystyle\sum_{m=1}^{\infty} \frac{(x+1)^m}{m^2}.$

9. $\displaystyle\sum_{m=1}^{\infty} \frac{[10(x+1)]^{2m}}{(m!)^2}.$

10. $\displaystyle\sum_{m=0}^{\infty} \frac{(9x)^{2m}}{(m!)^2}.$

11. $\displaystyle\sum_{m=0}^{\infty} \frac{2^m (x-2)^m}{10^m}.$

12. $\displaystyle\sum_{m=0}^{\infty} \frac{m! (x-2)^m}{10^m}.$

In Exercises 13–20, use a Taylor series found in this section to derive the Taylor series expansion of the given function at 0. In each case, specify the radius and interval of convergence.

13. $\dfrac{3-x}{1+x}.$

14. $\dfrac{x}{(1+x)^2}.$

15. $\dfrac{x}{(1+x^2)^2}.$

16. $\dfrac{x+2}{1-x^2}.$

17. $e^{3x^2+1}.$

18. $x\ln(1+x).$

19. $x \sin x \cos x.$

20. $x \sin(x+1).$

In Exercises 21-24, use the geometric series to derive the Taylor series expansion of the given function at 0. In each case, specify the radius and interval of convergence.

21. $\dfrac{1}{2+3x}$.   **22.** $\dfrac{1+x}{2-3x}$.   **23.** $3\dfrac{x+x^3}{2+3x^2}$.   **24.** $\dfrac{1}{x^2-2x+2}$

25. Let a be any real number $\neq 0$. Use ideas found in Example 7(a) to show that

$$\frac{1}{x} = \frac{1}{a}\sum_{m=0}^{\infty}(-1)^n\frac{(x-a)^n}{a^n},$$

with radius of convergence $R = |a|$. Thus $1/x$ is analytic everywhere except at 0.

26. Let

$$f(x) = \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{(k!)^2 2^{2k}} \quad \text{and} \quad g(x) = \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{k!(k+1)!2^{2k1}}.$$

Prove the following assertions. (a) Both power series converge for all x.

(b) $f'(x) = -g(x)$.   (c) $\displaystyle\int g(x)\,dx = f(x) + C$.   (d) $\displaystyle\int x\,f(x)\,dx = x\,g(x) + C$.

(The functions f and g are known as Bessel functions or order 0 and 1, respectively. They are usually denoted by $J_0(x)$ (for $f(x)$) and $J_1(x)$ (for $g(x)$). Bessel functions are studied in detail in Chapter 4.)

27. Let $f(x)$ be as in Exercise 26. (a) Find $f(0)$, $f'(0)$, $f''(0)$.
(b) More generally, find $f^{(k)}(0)$, for $k = 0, 1, 2, \ldots$.
(c) Integrate term-by-term to find $\int_0^1 f(x)\,dx$, then estimate the integral with a calculator.

28. Let $f(x) = \frac{1+x}{1-x}$. Find $f(0)$, $f'(0)$, and $f^{(100)}(0)$.

In Exercises 29-32, write the given expression as a single power series (and possibly some lower power terms, as in Example 8).

29. $\displaystyle\sum_{m=1}^{\infty}\frac{x^m}{m} - 2\sum_{m=0}^{\infty}mx^{m+1}$.

30. $\displaystyle\sum_{m=1}^{\infty}\frac{x^{m+1}}{m!} + \sum_{m=3}^{\infty}(m-1)x^{m-1}$.

31. $\displaystyle 2x\sum_{m=2}^{\infty}2\sqrt{m+2}\,x^m + \sum_{m=2}^{\infty}\frac{x^{m-1}}{m+6}$.

32. $\displaystyle (x+1)\sum_{m=2}^{\infty}2x^{m-1}$.

In Exercises 33-40, suppose that $y = \sum_{m=0}^{\infty}a_m x^m$ is a power series with a positive radius of convergence, and find a power series representation for the given expression.

33. $y' + y$.   **34.** $y' + x^2 y$.   **35.** $(1-x^2)y'' + 2xy'$.

36. $(2+x)y' + x^3 y$.   **37.** $x^2 y'' + y$.   **38.** $y'' + y' + xy$.

39. $xy' + y - e^x$.   **40.** $(1-x)y'' + \sin x$.

41. Geometric series. Fix a real (or complex) number x such that $|x| < 1$, and let n be a positive integer. (a) Show that

$$1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}.$$

(b) Conclude that the geometric series $\sum_{m=0}^{\infty} x^m$ converges to $1/(1 - x)$ if $|x| < 1$ and diverges otherwise. [Hint: What are the partial sums? Also, $x^{n+1} \to 0$ if and only if $|x| < 1$.]

42. Uniform convergence of the geometric series. (a) Let ρ be any number such that $0 < \rho < 1$. Use the Weierstrass M-test to show that the geometric series $\sum_{m=0}^{\infty} x^m$ converges uniformly for all x in $[-\rho, \rho]$.
(b) Show that the differentiated series $\sum_{m=1}^{\infty} mx^{m-1}$ converges uniformly on $[-\rho, \rho]$.
(c) Justify term-by-term differentiation of the geometric series. (See Theorem 5, Section 2.9.)

43. Uniform convergence of power series. Suppose that $\sum_{m=0}^{\infty} a_m x^m$ has radius of convergence $R > 0$. Let r be any real number such that $0 < r < R$.
(a) Show that $a_m r^m \to 0$. Conclude that $|a_m r^m| \leq M$ for some $M > 0$ and all m. [Hint: Apply the nth term test to a convergent series.]
(b) For x such that $|x| \leq r$, we have

$$|a_m x^m| = |a_m| r^m (|x|/r)^m \leq M(|x|/r)^m.$$

Use these inequalities, the fact that $|x|/r < 1$, and the Weierstrass M-test to conclude that $\sum_{m=0}^{\infty} a_m x^m$ converges uniformly on $[-r, r]$.

44. Term-by-term differentiation of power series. (a) Show that if $\sum_{m=0}^{\infty} a_m x^m$ has radius of convergence $R > 0$, then $\sum_{m=1}^{\infty} m a_m x^{m-1}$ has the same radius of convergence. [Hint: Given $0 \leq |x| < R$, let r be such that $|x| < r < R$. Then $|m a_m x^{m-1}| \leq m(|x|/r)^{m-1} |a_m| r^{m-1}$. Use the fact that the geometric series can be differentiated term-by-term (so $\sum m(|x|/r)^{m-1}$ is convergent); also prove and then use that $|a_m| r^{m-1} \leq M$ for some M.]
(b) Show that the derivative of $\sum_{m=0}^{\infty} a_m x^m$ is $\sum_{m=1}^{\infty} m a_m x^{m-1}$.
(c) Show that the derivative of $f(x) = \sum_{m=0}^{\infty} a_m (x - a)^m$ (for $|x - a| < R$) is $f'(x) = \sum_{m=1}^{\infty} m a_m (x - a)^{m-1}$ (for $|x - a| < R$). [Hint: Define $g(x) = f(x + a)$.]

A.5   The Power Series Method, Part II

The solutions of many differential equations have power series expansions. For example, the solution of $y' - y = 0$ is $y = Ce^x$ or $y = C \sum_{m=0}^{\infty} \frac{x^n}{n!}$. The power series method, that we present in this section, allows us to find the power series solution (when it exists) directly from the differential equation. This method is very useful in solving a wide class of important second-order linear differential equations with variable coefficients that cannot be solved by the methods that we discussed in previous sections. The power series method works as follows. First, assume that the solution is a power series. Second, write each term in the given differential equation as a power series. Third, equate coefficients of the resulting series on both sides of the equation.

Finally, solve for the unknown coefficients in the series representation of the assumed solution.

Let us illustrate this by deriving the power series solution of $y' - y = 0$.

EXAMPLE 1   A first-order equation

Assuming that you know that the solution of $y' - y = 0$ has a power series expansion about 0, find this solution.

Solution   Write $y = \sum_{m=0}^{\infty} a_m x^m$. Differentiating term-by-term we have $y' = \sum_{m=1}^{\infty} m a_m x^{m-1}$. If we substitute the power series of y and y' into the equation and combine the series, we obtain

To combine the series, we need to have the same powers of x appearing in both. For this purpose, we reindex the first series by changing m to $m+1$. The index of summation has to be lowered by 1, and so the first series will have to start at 0.

$$\sum_{m=1}^{\infty} m a_m x^{m-1} - \sum_{m=0}^{\infty} a_m x^m = 0;$$

$$\sum_{m=0}^{\infty} (m+1) a_{m+1} x^m - \sum_{m=0}^{\infty} a_m x^m = 0;$$

$$\sum_{m=0}^{\infty} [(m+1) a_{m+1} - a_m] x^m = 0.$$

By the identity principle, $(m+1)a_{m+1} - a_m = 0$ for all m, and so

$$(1) \qquad\qquad a_{m+1} = \frac{a_m}{m+1}.$$

Thus,

$$a_1 = a_0. \quad a_2 = \frac{a_1}{2} = \frac{a_0}{1 \cdot 2}, \quad a_3 = \frac{a_2}{3} = \frac{a_0}{1 \cdot 2 \cdot 3}, \quad a_4 = \frac{a_3}{4} = \frac{a_0}{1 \cdot 2 \cdot 3 \cdot 4}, \dots,$$

and in general $a_m = a_0/m!$. Hence the solution is

$$y = a_0 \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \right) = a_0 e^x,$$

where a_0 accounts for the arbitrary constant that we would expect in the general solution of a first-order linear differential equation.   ∎

Recall from the previous section that reindexing a series works like a change of variables in an integral. If k is a positive integer and you change m to $m + k$ inside the series, then the starting point of the series must be lowered by k. If you change m to $m - k$, then the starting point of the series must be raised by k.

An expression like (1) that gives each coefficient in terms of preceding ones is called a **recurrence relation**. In Example 1 the recurrence relation determines the coefficients in steps of one. That is, from a_0 we get a_1, from a_1 we get a_2, and so on. In the next example, the recurrence relation determines the coefficients in steps of two.

EXAMPLE 2   A familiar second-order equation

We know that two linearly independent solutions of the differential equation $y'' + y = 0$ are $y_1 = \cos x$ and $y_2 = \sin x$. We also know that both of these solutions have power series expansions about 0. Derive the power series solutions using the power series method.

Solution If we assume that the differential equation has a power series solution

$$y = \sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots ,$$

then the derivatives of y are obtained by term-by-term differentiation:

$$y' = \sum_{m=1}^{\infty} m\, a_m\, x^{m-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots ,$$

$$y'' = \sum_{m=2}^{\infty} m\,(m-1)\, a_m\, x^{m-2} = 2a_2 + 2 \cdot 3\, a_3\, x + 3 \cdot 4\, a_4 x^2 + \cdots .$$

Substituting y and y'' into the equation, we obtain

Change m to $m+2$ in the first series, and start the new series at 0.

$$\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} + \sum_{m=0}^{\infty} a_m x^m = 0,$$

$$\sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2}\, x^m + \sum_{m=0}^{\infty} a_m x^m = 0,$$

$$\sum_{m=0}^{\infty} [(m+2)(m+1)a_{m+2} + a_m]x^m = 0.$$

Thus

$$a_{m+2} = -\frac{1}{(m+2)(m+1)}\, a_m.$$

It is clear from this relation that a_0 determines a_2, which in turn determines a_4 and so on. Similarly, a_1 determines a_3, which determines a_5, and so on. Thus the even coefficients are determined from a_0 and the odd coefficients from a_1:

$$a_2 = -\frac{1}{2!}a_0, \qquad\qquad a_3 = -\frac{1}{3!}a_1,$$

$$a_4 = -\frac{1}{4 \cdot 3}a_2 = \frac{1}{4!}a_0, \qquad a_5 = -\frac{a_3}{5 \cdot 4} = \frac{1}{5!}a_1,$$

$$a_6 = -\frac{1}{6 \cdot 5}a_4 = -\frac{1}{6!}a_0, \quad a_7 = -\frac{a_5}{7 \cdot 6} = -\frac{1}{7!}a_1.$$

The pattern we see gives

$$a_{2k} = (-1)^k \frac{a_0}{(2k)!}, \quad a_{2k+1} = (-1)^k \frac{a_1}{(2k+1)!}.$$

Writing the even-indexed and the odd-indexed terms separately, we get the solution

$$y = a_0 \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} + a_1 \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}.$$

We recognize these series as the Taylor series for the sine and cosine; thus, as we already observed, $y = a_0 \cos x + a_1 \sin x$.   ∎

Note that a_0 and a_1 account for the two arbitrary constants that we expect in the general solution to a second-order differential equation.

In Examples 1 and 2 the solutions of the differential equations were familiar functions. The usefulness of the power series method, however, is more appreciated when we use it to find solutions that are not combinations of elementary functions. To apply the method, we need to know when does a solution have a power series expansion. Consider what happens if we assume that the solution of the differential equation $xy' + y = 0$ has a power series expansion at 0. Setting $y = \sum_{m=0}^{\infty} a_m x^m$, then

$$xy' + y = x \sum_{m=0}^{\infty} m a_m x^{m-1} + \sum_{m=0}^{\infty} a_m x^m = \sum_{m=0}^{\infty} (1+m) a_m x^m.$$

So $xy' + y = 0$ implies that $\sum_{m=0}^{\infty} (1+m) a_m x^m = 0$, which in turn implies that $(1+m) a_m = 0$ for all m. Since $1+m \neq 0$, it follows that $a_m = 0$, which implies that the solution is $y = 0$. But you can easily verify that $y = C/x$ is a solution (you can derive the solution by using Theorem 1, Appendix A.1, or by simply guessing it). Indeed, $y' = -C/x^2$ and so $xy' + y = (-C/x) + (C/x) = 0$. So the power series method did not work in this case. This is to be expected since the solution is not even defined at $x = 0$ and so it cannot possibly have a power series centered at 0.

Thus we cannot always assume that the solution has a power series expansion centered at a given point. The following theorem of Fuchs gives sufficient conditions that justify using the power series method.

**THEOREM 1
POWER SERIES
SOLUTIONS**

Suppose that the functions $p(x)$, $q(x)$, and $g(x)$ have power series expansions at $x = a$; then any solution of

$$(2) \qquad y'' + p(x)y' + q(x)y = g(x)$$

has a power series expansion at $x = a$. Thus any solution y can be expressed as a power series centered at a,

$$y = \sum_{m=0}^{\infty} a_m (x-a)^m.$$

Moreover, the radius of convergence of this power series is at least as large as the minimum of the radii of convergence of p, q, and g.

When $p(x)$, $q(x)$, and $g(x)$ have power series expansions with positive radius of convergence at $x = a$, the point a is called an **ordinary point** for the equation (2).

Since the existence of the solutions of (2) is guaranteed by Theorem 3, Appendix A.1, the real purpose of Theorem 1 is to address the existence of power series expansions of the solutions. For a proof of Theorem 1, see Chapter 3 of *Ordinary Differential Equations*. by Garrett Birkhoff and Gian-Carlo Rota, 2nd ed., Wiley, 1969. To appreciate the power of Theorem 1, note that it asserts that a complicated equation such as $y'' + e^x y' + \cos xy = 1/(1+x^2)$ has a power series solution about $x = 0$ with radius of convergence at least 1.

EXAMPLE 3   A second-order initial value problem
Solve the initial value problem

$$y'' + xy' + y = 0, \quad y(0) : 0, \quad y'(0) = 1.$$

Solution Using the notation of Theorem 1, we let $p(x) = x$ and $q(x) = 1$. Since both functions have power series expansions at 0 (given by the functions themselves), we can assume a solution of the form

$$y = \sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + \cdots .$$

At the outset, we may use the initial conditions to find a_0 and a_1. We have

$$y(0) = a_0 + a_1 0 + a_2 0^2 + \cdots = a_0$$

$$y'(0) = a_1 + 2\, a_2 0 + 3\, a_3 0^2 + \cdots = a_1$$

and so the initial conditions imply that $a_0 = 0$ and $a_1 = 1$. Substituting the series for y and its derivatives into the differential equation yields

$$\sum_{m=2}^{\infty} m(m-1)\, a_m\, x^{m-2} + \sum_{m=1}^{\infty} ma_m x^m + \sum_{m=0}^{\infty} a_m x^m = 0.$$

Shifting the index of summation in the first series yields

$$\sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2}x^m + \sum_{m=1}^{\infty} ma_m x^m + \sum_{m=0}^{\infty} a_m x^m = 0;$$

$$(2a_2 + a_0) + \sum_{m=1}^{\infty} [(m+2)(m+1)\, a_{m+2} + ma_m + a_m]x^m = 0.$$

Equating the constant term to zero gives $a_2 = -a_0/2$ and equating the coefficients of the series to zero yields the recurrence relation

$$a_{m+2} = \frac{-1}{m+2}\, a_m.$$

As in the preceding example, a_0 determines the coefficients with even indices and a_1 those with odd indices. Since $a_0 = 0$, we get $a_2 = 0$, $a_4 = 0$, and so on. For

the coefficients with odd indices. the recurrence relation gives

$$a_1 = 1,$$

$$a_3 = -\tfrac{1}{3}a_1 = -\tfrac{1}{3},$$

$$a_5 = -\tfrac{1}{5}a_3 = \tfrac{1}{5 \cdot 3},$$

$$a_7 = -\tfrac{1}{7}a_5 = -\tfrac{1}{7 \cdot 5 \cdot 3}.$$

The pattern that emerges is

$$a_{2k+1} = \frac{(-1)^k}{1 \cdot 3 \cdot 5 \cdots (2k+1)}.$$

Substituting these values for the coefficients into y, we get the solution to the given initial value problem

$$y = x - \frac{1}{3}x^3 + \frac{1}{15}x^5 - \frac{1}{105}x^7 + \cdots + \frac{(-1)^k}{1 \cdot 3 \cdot 5 \cdots (2k+1)}x^{2k+1} + \cdots .$$

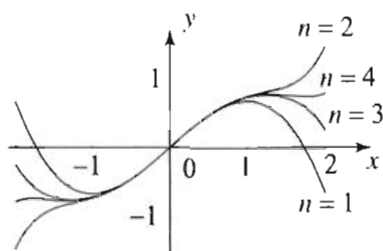This solution can be expressed in a more compact form (see Exercise 23).   ∎

Figure 1 Polynomial (degree $2n + 1$) approximation of the solution in Example 3.

Polynomial approximations of the solution of Example 3 are graphed in Figure 1. The more terms we include, the better the approximation is on a larger interval. Note that all the polynomials satisfy the initial conditions.

It is useful to remark that when solving initial value problems like Example 3, using the power series method, we always have

$$y(0) = a_0 \quad \text{and} \quad y'(0) = a_1.$$

As should be expected, it is not always possible to find simple patterns for the coefficients in the solution. In such cases we can determine as many terms of the series as we wish in order to obtain a Taylor polynomial approximation of the solution.

EXAMPLE 4   Approximation by a Taylor polynomial
Find a fourth degree Taylor polynomial approximation of the solution of the initial value problem

$$y'' + e^x y = \cos x, \quad y(0) = 1, \quad y'(0) = 0.$$

Solution In the notation of Theorem 1, we have $p(x) = 0$, $q(x) = e^x$, and $g(x) = \cos x$, and each of these functions has a power series expansion at 0, valid for all x. Hence we have a power series solution, centered at 0 with an infinite radius of convergence. Let us write this series as a Taylor series in the form

$$y = \sum_{m=0}^{\infty} \frac{y^{(m)}(0)}{m!}x^m.$$

Our goal is to determine the first five coefficients, $a_0, a_1, \ldots, a_5$, in order to obtain a Taylor polynomial approximation

$$P_4(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y^{(4)}(0)}{4!}x^4.$$

Instead of using the power series method, we will use the differential equation and the initial conditions directly. Indeed the initial conditions, $y(0) = 1$ and $y'(0) = 0$, give us immediately the first two coefficients of the Taylor polynomial. From the differential equation, we have

$$
\begin{aligned}
y'' &= -e^x y + \cos x. \\
y''(0) &= -e^0 y(0) + \cos(0) = (-1)(1) + 1 = 0; \\
y''' &= \left[-e^x y + \cos x \right]' = -e^x y - e^x y' - \sin x, \\
y'''(0) &= -e^0 y(0) - e^0 y'(0) - \sin(0) = -1; \\
y^{(4)} &= \left[-e^x(y + y') - \sin x \right]' = -e^x(y + y') - e^x(y' + y'') - \cos x, \\
y^{(4)}(0) &= -1 - 1 = -2.
\end{aligned}
$$

Thus $P_4(x) = 1 - (1/3!)x^3 - (2/4!)x^4 = 1 - x^3/6 - x^4/12$ is the fourth-degree Taylor polynomial approximation of the solution.

We can continue this procedure and find as many terms as we please. Here is the seventh-degree Taylor polynomial approximation

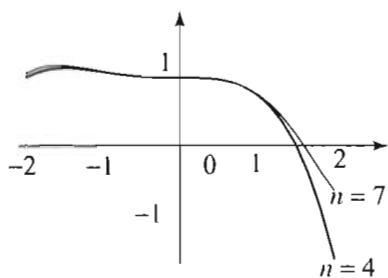$$P_7(x) = 1 - \frac{1}{6}x^3 - \frac{1}{12}x^4 + \frac{1}{120}x^6 + \frac{19}{5,040}x^7.$$

On the interval $[-1, 1]$, the graph of the approximating polynomial of degree 4 is indistinguishable from the graph of the polynomial of degree 7 (Figure 2). This suggests that on this interval we have a good approximation with a polynomial of degree 4. However, as we move away from the center, more terms are needed to get a good approximation. ∎

Figure 2 Polynomial (degree n) approximation of the solution.

In second-order homogeneous problems, it is instructive to identify linearly independent solutions y_1 and y_2. As we saw in Examples 2 and 3. the recurrence relations allowed us to solve for the coefficients a_m, $m \geq 2$, in terms of a_0 and a_1. We can define y_1 by taking $a_0 = 1$, $a_1 = 0$ and y_2 by taking $a_0 = 0$, $a_1 = 1$. Then the general solution can be written as

$$y = c_1 y_1 + c_2 y_2.$$

In terms of initial value problems, it is easily seen that y_1 satisfies the initial values $y_1(a) = 1$, $y_1'(a) = 0$. Similarly, y_2 satisfies $y_2(a) = 0$, $y_2'(a) = 1$.

The following example deals with a particular case of a Legendre's differential equation. A detailed treatment of these equations is presented in Sections 5.5 and 5.6. This treatment is one of the most important applications of the power series method in this book.

EXAMPLE 5   A Legendre's differential equation
Solve the initial value problem

$$(1 - x^2)y'' - 2xy' + 12y = 0, \quad y(0) = 0, \quad y'(0) = 1.$$

Solution Put the equation in standard form:

$$y'' - \frac{2x}{1 - x^2}\, y' + \frac{12}{1 - x^2}\, y = 0.$$

In the notation of Theorem 1, we have $p(x) = -2x/(1-x^2)$ and $q(x) = 12/(1-x^2)$. From the geometric series, $\sum_{m=0}^{\infty} x^m = 1/(1-x)$, which is valid for $|x| < 1$, we obtain

$$\frac{1}{1 - x^2} = \sum_{m=0}^{\infty} x^{2m} \quad \text{and} \quad \frac{x}{1 - x^2} = \sum_{m=0}^{\infty} x^{2m+1},$$

where both series converge for all $|x^2| < 1$ or $|x| < 1$. From this it follows that $p(x)$ and $q(x)$ have power series expansions about 0 with radius of convergence equal to 1. Theorem 1 guarantees that the solution has a power series expansion at 0 with radius of convergence at least equal to 1. To find this solution, plug $y = \sum_{m=0}^{\infty} a_m x^m$ and its first and second derivatives into the equation and get

$$(1 - x^2) \sum_{m=2}^{\infty} m(m - 1)a_m x^{m-2} - 2x \sum_{m=1}^{\infty} m a_m x^{m-1} + 12 \sum_{m=0}^{\infty} a_m x^m = 0;$$

$$\sum_{m=2}^{\infty} m(m - 1)a_m x^{m-2} - \sum_{m=2}^{\infty} m(m - 1)a_m x^m$$

$$- \sum_{m=1}^{\infty} 2m a_m x^m + 12 \sum_{m=0}^{\infty} a_m x^m = 0.$$

Changing m to $m + 2$ in the first series in the second equation and simplifying,

$$\sum_{m=0}^{\infty} (m + 2)(m + 1)a_{m+2} x^m - \sum_{m=1}^{\infty} m(m - 1)a_m x^m$$

$$- \sum_{m=1}^{\infty} 2m a_m x^m + 12 \sum_{m=0}^{\infty} a_m x^m = 0;$$

$$2a_2 + 12a_0 + \sum_{m=1}^{\infty} \left[(m + 2)(m + 1)a_{m+2} + a_m \overbrace{\left(-m(m - 1) - 2m + 12\right)}^{(m-3)(m+4)}\right]x^m,$$

where we have split off the $m = 0$ terms from the series in order to start all of them at $m = 1$. Setting the coefficients in the series equal to 0 and simplifying, we obtain the relations

$$2a_2 + 12a_0 = 0; \quad a_{m+2} = a_m \frac{(m - 3)(m + 4)}{(m + 2)(m + 1)} \quad (m \geq 1).$$

The initial conditions determine a_0 and a_1. Indeed, $a_0 = y(0) = 0$ and $a_1 = y'(0) = 1$. The two step-recurrence relation determines the remaining coefficients. From

$a_0 = 0$, we get that $a_2 = 0$, $a_4 = 0$, and, more generally, $a_{2k} = 0$ for all $k \geq 0$. We now turn to the odd indices. Taking $m = 1$ in the recurrence relation, we obtain: $a_3 = a_1(-10)/6 = -5/3$. Next, we take $m = 3$ and find $a_5 = 0$, and so $a_7 = 0$, $a_9 = 0$, and, more generally, $a_{2k+1} = 0$ for all $k \geq 1$. Thus the solution is a polynomial:

$$y = a_1 x + a_3 x^3 = x - \frac{5}{3}x^3.$$

This can be verified directly from the equation: $y' = 1 - 5x^2$, $y'' = -10x$, so

$$(1 - x^2)y'' - 2xy' + 12y = (1 - x^2)(-10x) - 2x(1 - 5x^2) + 12\left(x - \frac{5}{3}x^3\right) = 0.$$

The initial conditions are obviously verified.   ∎

Recall that Theorem 1 guarantees that the power series solution will have a radius of convergence at least as large as the smallest of the radii of convergence of the power series expansions of the coefficient functions $p(x)$ and $q(x)$. Keep in mind, though, that the solution may have a larger radius of convergence. In Example 5 one solution is a polynomial of degree 3; in particular, its radius of convergence is infinite.

We next derive the general solution of the equation in Example 5.

EXAMPLE 6   General solution of a Legendre's differential equation

Find two linearly independent solutions of Legendre's equation

$$(1 - x^2)y'' - 2xy' + 12y = 0.$$

Solution In Example 5 we found one particular solution, $y_1 = x - (5/3)x^3$. To find a second linearly independent solution, we return to the recurrence relation that we found in the previous solution and let a_0 be arbitrary. This gives $a_2 = a_0 \frac{(-3)(4)}{(2)(1)} = -6 a_0$. $a_4 = a_2 \frac{(-1)(6)}{(4)(3)} = 3 a_0$, $a_6 = a_4 \frac{(1)(8)}{(6)(5)} = \frac{4}{5} a_0$, and so on. Thus,

$$y_2 = a_0\left(1 - 6x^2 + 3x^4 + \frac{4}{5}x^6 + \cdots\right).$$

The solution is an infinite power series in even powers of x. Unlike y_1, y_2 has infinitely many nonzero terms.   ∎

The fact that Legendre's equation in Example 5 has a polynomial solution and an infinite series solution is not a coincidence. Indeed this will happen every time the equation can be put in the form $(1 - x^2)y'' - 2xy' + n(n+1)y = 0$ for some nonnegative integer n. (In Example 5, we have $n = 3$.) Many interesting facts about Legendre's equation can be found in Sections 5.5 and 5.6.

Exercises A.5

*In Exercises 1 12, show that 0 is an ordinary point, and then find the power se-
ries solutions centered at 0. In each case, write down the recurrence relation for
the coefficients, and compute at least three nonzero terms from each power series
solution.*

1. $y' + 2xy = 0$.

2. $y' + y = 0$.

3. $y' + y = x$.

4. $y' + (\cos x)y = 0$.

5. $y'' - y = 0$.

6. $y'' - y = x$.

7. $y'' - x y' + y = 0$.

8. $y'' + 2 y' + 2 y = 0$.

9. $y'' + 2 x y' + y = 0$.

10. $y'' + x y' + y = 0$.

11. $y'' - 2 y' + y = 0$, $\quad y(0) = 0$, $y'(0) = 1$.

12. $y'' - 2y' + y = x$, $\quad y(0) = 2, y'(0) = 1$.

*In Exercises 13–16, use the power series method to solve the given Legendre's dif-
ferential equation.*

13. $(1 - x^2)y'' - 2xy' + 2y = 0$, $\ y(0) = 0$, $\ y'(0) = 3$.

14. $(1 - x^2)y'' - 2xy' + 6y = 0$, $\ y(0) = 1$, $\ y'(0) = 1$.

15. $(1 - x^2)y'' - 2xy' + 12y = 0$, $\ y(0) = 1$, $\ y'(0) = 0$.

16. $(1 - x^2)y'' - 2xy' + 2y = 0$, $\ y(0) = 1$, $\ y'(0) = 0$.

17. Find two linearly independent solutions of $(1 - x^2)y'' - 2xy' + 2y = 0$ on the
interval $-1 < x < 1$. [Hint: From the power series method, one solution is a first
degree polynomial. You may use the reduction of order formula to find a second
linearly independent solution.]

18. Find two linearly independent solutions of $(1 - x^2)y'' - 2xy' + 6y = 0$ on the
interval $-1 < x < 1$.

*In Exercises 19–22, (a) justify the existence of a power series solution centered at
0, and then find a fifth-degree Taylor polynomial approximation of the solution. (b)
With the help of a computer, find higher-degree Taylor polynomials that approx-
imate the solution. Plot them on the interval $[-2, 2]$, and then decide the lowest
degree needed in order to obtain a good approximation of the solution on the interval
$[-1, 1]$.*

19. $y'' - y' + 2y = e^x$, $\ y(0) = 0$, $\ y'(0) = 1$.

20. $y'' + (\sin x)y = x$, $\ y(0) = 0, y'(0) = 1$.

21. $y'' + e^x y = 0$, $\ y(0) = 1, y'(0) = 0$.

22. $y'' + x y = \cosh x$, $\ y(0) = 0, y'(0) = 1$.

23. Show that the solution in Example 3 is

$$y = \sum_{k=0}^{\infty} \frac{(-1)^k 2^k k!}{(2k + 1)!} x^{2k+1}.$$

24. Airy's differential equation. In this exercise, we find a series solution of
Airy's equation

$$y'' - xy = 0.$$

(a) State why the solutions of the differential equation have power series solutions with infinite radius of convergence.

(b) Assume that $y = \sum_{m=0}^{\infty} a_m x^m$. Show that $a_2 = 0$ and derive the *three-step recurrence relation*

$$(m + 2)(m + 1)a_{m+2} = a_{m-1}, \qquad m = 1, 2, 3, \ldots .$$

(c) Show that the general solution is

$$y \;=\; a_0 \left[1 + \sum_{n=1}^{\infty} \frac{x^{3n}}{(3n)(3n - 1)(3n - 3) \ldots 3 \cdot 2} \right]$$

$$+ a_1 \left[x + \sum_{n=1}^{\infty} \frac{x^{3n+1}}{(3n + 1)(3n)(3n - 2) \ldots 4 \cdot 3} \right].$$

(d) Write $y = a_0 y_1 + a_1 y_2$ and plot several polynomial approximations of y_1 and y_2. Note that for negative values of x the graphs look like those of $\sin x$ and $\cos x$, and for positive values of x the graphs look like those of e^x. Examine the differential equation and give a reason for these observations.

25. Solve the following initial value problem for Airy's equation:

$$y'' - xy = 0, \quad y(0) = 1, \; y'(0) = 0.$$

26. The sine and cosine functions. In this exercise, we show how a differential equation can be used to derive properties of its solutions. We refer to the differential equation of Example 2 with the two linearly independent solutions

$$C(x) = \sum_{k=0}^{\infty}(-1)^k \frac{1}{(2k)!}x^{2k} \quad \text{and} \quad S(x) = \sum_{k=0}^{\infty}(-1)^k \frac{1}{(2k + 1)!}x^{2k+1}.$$

We pretend for now that we do not recognize these functions as the sine and cosine functions, and we derive their basic properties from the differential equation.

(a) Show that $C'(x) = -S(x)$ and $S'(x) = C(x)$.

(b) Using Abel's form of the Wronskian, show that $C^2(x) + S^2(x) = 1$.

(c) Using the fact that any solution of the differential equation $y'' + y = 0$ is a linear combination of $C(x)$ and $S(x)$, show that $S(x + a) = S(x)C(a) + C(x)S(a)$. [Hint: Show that $S(x + a)$ is a solution and conclude that $S(x + a) = c_1 S(x) + c_2 C(x)$. Determine the constants by evaluating at $x = 0$.]

(d) Establish the identity

$$C(x + a) = C(x)C(a) - S(x)S(a).$$

(e) Which trigonometric identities are implied by (c) and (d)?

A.6   The Method of Frobenius

While the method of power series is useful in many situations, and perhaps the most widely applicable method that we have developed thus far, there are still important differential equations that cannot be fully understood by this method. For example, consider Bessel's equation

$$x^2 y'' + xy' + (x^2 - p^2)y = 0.$$

Putting it in standard form, we see that $x = 0$ is not an ordinary point. Hence, Theorem 1 of Appendix A.5 does not apply. For applications, it is of particular importance to understand the behavior of the solutions at $x = 0$. To this end, we will develop a generalization of the power series method, known as the **method of Frobenius**.

Consider the homogeneous differential equation

(1) $$y'' + p(x)y' + q(x)y = 0.$$

Recall that a is an ordinary point of the differential equation if p and q have power series expansions at a. Otherwise, a is called a **singular point**. If a is an ordinary point, then we know from our work in the preceding section that the solutions of this equation have power series expansions at the point $x = a$. If a is a singular point, then the method of power series fails, as our next example shows.

EXAMPLE 1   An equation for which the power series method fails
Consider the differential equation $xy'' + 2y' + xy = 0$.
(a) Determine whether $x = 0$ is an ordinary or singular point.
(b) Apply the method of power series at $x = 0$ to the differential equation.
Solution (a) We have $p(x) = 2/x$, and $q(x) = 1$. Since $p(x)$ does not have a power series expansion at $x = 0$, the point $x = 0$ is a singular point of the equation.
(b) We assume that y has a power series representation centered at $x = 0$, $y = \sum_{m=0}^{\infty} a_m x^m$. Substituting into the differential equation and reindexing as needed gives

$$\sum_{m=2}^{\infty} m(m-1)a_m x^{m-1} + \sum_{m=1}^{\infty} 2 m a_m x^{m-1} + \sum_{m=0}^{\infty} a_m x^{m+1} = 0;$$

$$\sum_{m=1}^{\infty} (m+1)m a_{m+1} x^{m} + \sum_{m=0}^{\infty} 2(m+1)a_{m+1} x^{m} + \sum_{m=1}^{\infty} a_{m-1} x^{m} = 0;$$

$$2a_1 + \sum_{m=1}^{\infty} [m(m+1)a_{m+1} + 2(m+1)a_{m+1} + a_{m-1}] x^{m} = 0.$$

Equating coefficients, we find that $a_1 = 0$ and

$$m(m+1)a_{m+1} + 2(m+1)a_{m+1} + a_{m-1} = 0; \quad \text{for all } m,$$

from which we get the two-step recurrence relation

$$a_{m+1} = -\frac{1}{(m+2)(m+1)}a_{m-1}.$$

Since $a_1 = 0$, it follows that $a_3 = a_5 = a_7 = \cdots = 0$.   For the even indexed coefficients we have

$$a_2 = -\tfrac{1}{3\cdot 2}a_0 = -\tfrac{1}{3!}a_0$$

$$a_4 = -\tfrac{1}{5\cdot 4}a_2 = \tfrac{1}{5!}a_0$$

$$a_6 = -\tfrac{1}{7\cdot 6}a_4 = -\tfrac{1}{7!}a_0.$$

It is clear how this sequence continues. Thus one solution is

$$y = a_0\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots\right) = \frac{a_0}{x}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right) = a_0\frac{\sin x}{x}$$
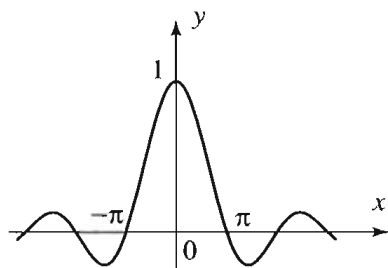

Figure 1 Graph of $y = \frac{\sin x}{x}$.

(Figure 1).

(Note that the function $\frac{\sin x}{x}$ has a power series expansion at $x = 0$ that converges for all x.) Since the differential equation is of second order, we need two linearly independent solutions. The method of power series yielded only one solution and hence failed to give the general solution. To get a second linearly independent solution, we can use the method of reduction of order, which gives (you should check this)

$$y = \frac{\cos x}{x}.$$

(See also Exercise 29.)   ∎

Put simply, the method of power series failed to generate both linearly independent solutions in Example 1 because one of those solutions is not a power series. To see why, we write

$$\frac{\cos x}{x} = x^{-1}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right).$$

This is not a power series, since it contains a negative power of x. It turns out that if the singular point of a differential equation is not "very bad" then the solutions follow a pattern similar to that of Example 1. That is, at least one solution is a power series multiplied by x^r. We devote the rest of this section to making this idea precise. For simplicity, we consider only series expansions about $x = 0$, although analogous methods apply for series expanded about an arbitrary $x = a$.

Suppose that $x = 0$ is a singular point of the equation

(2) $$y'' + p(x)y' + q(x)y = 0.$$

We say that $x = 0$ is a **regular singular point** of the equation if both of the functions $xp(x)$ and $x^2q(x)$ have power series expansions at $x = 0$. (This

is what we mean by a singularity that is not "very bad.")   The Frobenius method that we now describe applies to equations for which $x = 0$ is a regular singular point.   For clarity's sake, we restrict our attention to the case $x > 0$.   The case $x < 0$ is handled similarly (or can be reduced to the case $x > 0$ by the change of variables $t = -x$).   Motivated by Example 1, we try a series solution of the form

$$
(3) \qquad y = x^r \left(a_0 + a_1 x + a_2 x^2 + \cdots \right) = \sum_{m=0}^{\infty} a_m x^{r+m}
$$

with $a_0 \neq 0$. Such a series is called a **Frobenius series**. We have

$$
y' = \sum_{m=0}^{\infty} a_m (r + m) x^{r+m-1}
$$

and

$$
y'' = \sum_{m=0}^{\infty} a_m (r + m)(r + m - 1) x^{r+m-2}.
$$

(Observe that the index of summation begins at 0 in both these series. Why?) Thus (2) becomes

$$
\sum_{m=0}^{\infty} a_m (r + m)(r + m - 1) x^{r+m-2}
$$
$$
+ p(x) \sum_{m=0}^{\infty} a_m (r + m) x^{r+m-1} + q(x) \sum_{m=0}^{\infty} a_m x^{r+m} = 0.
$$

We factor x from the second series and x^2 from the third to make all exponents the same and get

$$
(4) \quad \sum_{m=0}^{\infty} a_m (r + m)(r + m - 1) x^{r+m-2}
$$
$$
+ x p(x) \sum_{m=0}^{\infty} a_m (r + m) x^{r+m-2} + x^2 q(x) \sum_{m=0}^{\infty} a_m x^{r+m-2} = 0.
$$

Since by assumption $x = 0$ is a regular singular point, the functions $x p(x)$ and $x^2 q(x)$ have power series expansions about 0, say

$$
x p(x) = p_0 + p_1 x + p_2 x^2 + \cdots
$$

and

$$
x^2 q(x) = q_0 + q_1 x + q_2 x^2 + \cdots .
$$

Substituting these into (4) gives

$$\sum_{m=0}^{\infty} a_m (r+m)(r+m-1)x^{r+m-2}$$

$$+(p_0 + p_1 x + p_2 x^2 + \cdots) \sum_{m=0}^{\infty} a_m (r+m)x^{r+m-2}$$

$$+(q_0 + q_1 x + q_2 x^2 + \cdots) \sum_{m=0}^{\infty} a_m x^{r+m-2} = 0.$$

The total coefficient of each power of x on the left side of this equation must be 0, since the right side is 0. The lowest power of x that appears in the equation is x^{r-2}. Its coefficient is $a_0 r(r-1) + p_0 a_0 r + q_0 a_0 = a_0[r(r-1) + p_0 r + q_0] = 0$. Since $a_0 \neq 0$, r must be a root of the **indicial equation**

(5)
$$\boxed{r(r-1) + p_0 r + q_0 = 0.}$$

The roots of this equation are called the **indicial roots** and are denoted by r_1 and r_2 with the convention that $r_1 \geq r_2$ whenever they are real. Note that p_0 and q_0 are easily determined, since they are the values of $xp(x)$ and $x^2 q(x)$ at $x = 0$. Once we have determined r_1 and r_2, we substitute r_1 in (4) and solve for the unknown coefficients a_n as we would do with the power series method. This will determine a first solution of (2). Summing up, we have the following important result.

THEOREM 1 FROBENIUS METHOD FIRST SOLUTION

If $x = 0$ is a regular singular point of the equation

$$y'' + p(x)y' + q(x)y = 0,$$

then one solution is of the form

$$y_1 = |x|^{r_1} \left(a_0 + a_1 x + a_2 x^2 + \cdots \right), \qquad a_0 \neq 0,$$

where r_1 is a root of the indicial equation (5), with the convention that r_1 is the larger of the two roots when both roots are real.

For a proof of this theorem, see Chapter 9 of *Ordinary Differential Equations* by Garrett Birkhoff and Gian-Carlo Rota, 2nd ed., Wiley, 1969. It can be shown that if $xp(x)$ and $x^2 q(x)$ have power series expansions that converge for $|x| < R$, then the series solution will converge for $0 < |x| < R$.

Note that Theorem 1 says nothing about a second solution of (2). This solution may have one of three forms, as described in the following box. The proofs are presented in an appendix at the end of this section.

THEOREM 2
THE FROBENIUS
METHOD

Suppose that $x = 0$ is a regular singular point of the differential equation

$$y'' + p(x)y' + q(x)y = 0,$$

and let r_1 and r_2 denote the indicial roots.   The differential equation has two linearly independent solutions y_1 and y_2, as we now describe.

Case 1.   If $r_1 - r_2$ is not an integer, then

$$y_1 = |x|^{r_1} \sum_{m=0}^{\infty} a_m x^m, \quad y_2 = |x|^{r_2} \sum_{m=0}^{\infty} b_m x^m, \quad a_0 \neq 0 \text{ and } b_0 \neq 0.$$

Case 2.   If $r = r_1 = r_2$, then

$$y_1 = |x|^{r} \sum_{m=0}^{\infty} a_m x^m, \quad y_2 = y_1 \ln|x| + |x|^{r} \sum_{m=1}^{\infty} b_m x^m, \quad a_0 \neq 0.$$

Case 3.   If $r_1 - r_2$ is a positive integer, with $r_1 \geq r_2$, then

$$y_1 = |x|^{r_1} \sum_{m=0}^{\infty} a_m x^m, \quad y_2 = k y_1 \ln|x| + |x|^{r_2} \sum_{m=0}^{\infty} b_m x^m.$$

where $a_0 \neq 0$, $b_0 \neq 0$ (k may or may not be 0).

Thus we find two linearly independent Frobenius series solutions in Case 1 and also in Case 3 when $k = 0$.   Otherwise. we can find only one Frobenius series solution, and any linearly independent solution must involve a logarithmic term.   In computing y_1 and y_2 it is convenient to take $a_0 = 1$ and $b_0 = 1$; then the two arbitrary constants will appear when writing the general solution in the form of an arbitrary linear combination of y_1 and y_2.

EXAMPLE 2   Frobenius method: $r_1 - r_2$ is not an integer
Solve the differential equation

$$y'' + \frac{1}{2x}y' - \frac{1}{4x}y = 0, \quad x > 0.$$

Solution   Theorem 1 applies, since both $xp(x) = \frac{1}{2}$ and $x^2 q(x) = -\frac{1}{4}x$ have power series expansions at $x = 0$.   Moreover, $p_0 = \frac{1}{2}$ and $q_0 = 0$, so the indicial equation and indicial roots are

$$r(r-1) + \tfrac{1}{2}r = 0.$$
$$r^2 - \tfrac{1}{2}r = 0,$$
$$r_1 = \tfrac{1}{2} \quad \text{and} \quad r_2 = 0.$$

(Following our convention, we choose $r_1 \geq r_2$.)   Since $r_1 - r_2$ is not an integer, the solutions are given by Case I of the Frobenius method.   The next step is to

determine the unknown coefficients in the series solutions. It is more convenient to work with the differential equation if we multiply both sides by $4x$:

$$4xy'' + 2y' - y = 0.$$

Substituting $y = \sum_{m=0}^{\infty} a_m x^{m+r}$ into the equation gives

$$\sum_{m=0}^{\infty} 4a_m(m+r)(m+r-1)x^{m+r-1}$$

$$+ \sum_{m=0}^{\infty} 2a_m(m+r)x^{m+r-1} - \sum_{m=1}^{\infty} a_{m-1}x^{m+r-1} = 0.$$

(We have shifted the index in the third series to match all the exponents of x.) Adding the terms corresponding to $m = 0$ and setting the coefficient of x^{0+r-1} equal to 0 gives us the indicial equation

$$4a_0 r(r-1) + 2a_0 r = 0,$$

$$r^2 - \frac{1}{2}r = 0 \quad (\text{since } a_0 \neq 0).$$

Setting the sum of the coefficients of x^{m+r-1} equal to 0 gives us

(6) $$4a_m(m+r)(m+r-1) + 2a_m(m+r) - a_{m-1} = 0.$$

At this point, we treat separately the case $r_1 = \frac{1}{2}$ and $r_2 = 0$. When $r = \frac{1}{2}$, the recurrence relation (6) becomes

$$4a_m(m + \frac{1}{2})(m + \frac{1}{2} - 1) + 2a_m(m + \frac{1}{2}) - a_{m-1} = 0$$

$$a_m = \frac{1}{2m(2m+1)}a_{m-1}.$$

From this recurrence relation we get the pattern

$$a_1 = \frac{1}{2 \cdot 3}a_0 = \frac{1}{3!}a_0, \quad a_4 = \frac{1}{8 \cdot 9}a_3 = \frac{1}{9!}a_0,$$

$$a_2 = \frac{1}{4 \cdot 5}a_1 = \frac{1}{5!}a_0, \quad a_5 = \frac{1}{10 \cdot 11}a_4 = \frac{1}{11!}a_0,$$

$$a_3 = \frac{1}{6 \cdot 7}a_2 = \frac{1}{7!}a_0, \qquad \vdots$$

Thus $a_m = \frac{1}{(2m+1)!}a_0$, and hence the solution corresponding to $r = \frac{1}{2}$ and $a_0 = 1$ is

$$y_1 = \sum_{m=0}^{\infty} \frac{1}{(2m+1)!} x^{m+\frac{1}{2}}.$$

For $r = 0$ we get the recurrence relation (6):

$$4a_m m(m-1) + 2a_m m - a_{m-1} = 0$$

$$a_m = \frac{1}{2m(2m-1)}a_{m-1}.$$

This gives the following pattern for the coefficients:

$$a_1 = \frac{1}{2 \cdot 1} a_0 = \frac{1}{2!} a_0, \qquad a_4 = \frac{1}{8 \cdot 7} a_3 = \frac{1}{8!} a_0,$$

$$a_2 = \frac{1}{4 \cdot 3} a_1 = \frac{1}{4!} a_0, \qquad a_5 = \frac{1}{10 \cdot 9} a_4 = \frac{1}{10!} a_0,$$

$$a_3 = \frac{1}{6 \cdot 5} a_2 = \frac{1}{6!} a_0, \qquad \vdots$$

It is clear that $a_m = \frac{1}{(2m)!} a_0$, and hence the solution corresponding to $r = 0$ and $a_0 = 1$ is

$$y_2 = \sum_{m=0}^{\infty} \frac{1}{(2m)!} x^m.$$

Thus the general solution of the differential equation is $y = c_1 y_1 + c_2 y_2$, where c_1 and c_2 are arbitrary constants. ∎

The solutions in Example 2 can be simplified as

$$y_1 = \sinh \sqrt{x} \quad \text{and} \quad y_2 = \cosh \sqrt{x}, \quad x > 0.$$

Figure 2 Solutions and partial sum approximation from Example 2.

Do you see why? [Hint: Write x^m as $(\sqrt{x})^{2m}$.] We can verify the validity of these solutions by plugging them into the differential equation. In Figure 2 we compare the graphs $\cosh \sqrt{x}$ and the partial sums of y_2. Notice how the partial sums of y_2 approximate the real solution. With the aid of a computer system, you can do the same for y_1.
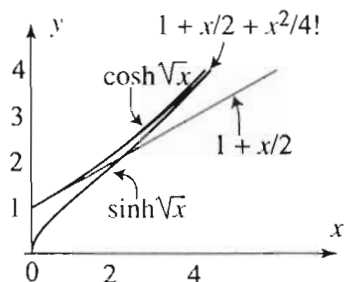

EXAMPLE 3   Case of a double root $r = r_1 = r_2$
Solve the differential equation

(7) $$x^2 y'' + 3xy' + (1 - x)y = 0, \quad x > 0.$$

Solution   We easily verify that $p_0 = 3$, $q_0 = 1$, and so the indicial equation is $r^2 + 2r + 1 = 0$. We have a double root $r = -1$, and so we are in Case 2. Let $y_1 = \sum_{m=0}^{\infty} a_m x^{m-1}$ with $a_0 \neq 0$. Substituting in (7), after some computations as in Example 2, we arrive at the recurrence relation

$$a_{m+1} = \frac{a_m}{(m+1)^2}.$$

Taking $a_0 = 1$, we get the first solution

$$y_1 = \sum_{m=0}^{\infty} \frac{1}{(m!)^2} x^{m-1} = \frac{1}{x}\left(1 + x + \frac{x^2}{4} + \frac{x^3}{36} + \frac{x^4}{576} + \cdots\right).$$
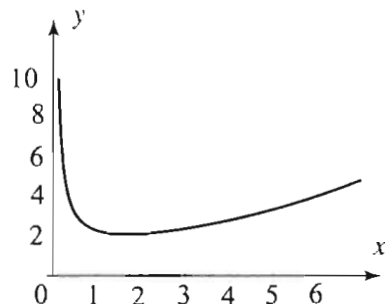

Figure 3 Approximation of y_1 near 0. Notice the blow-up due to the term $\frac{1}{x}$.

(An approximation of y_1 is shown in Figure 3, using a partial sum of the infinite series. The details of the derivation of y_1 are left to Exercise 30.) According to Case 2, we try for a second solution

$$y = y_1 \ln x + \sum_{m=1}^{\infty} b_m x^{m-1}.$$

Substituting in (7) and simplifying, we get

(8)
$$\overbrace{[x^2 y_1'' + 3xy_1' + (1-x)y_1]}^{=0} \ln x + (2xy_1' + 2y_1)$$
$$+ \sum_{m=1}^{\infty} b_m(m-1)(m-2)x^{m-1} + \sum_{m=1}^{\infty} 3b_m(m-1)x^{m-1}$$
$$+ \sum_{m=1}^{\infty} b_m x^{m-1} - \sum_{m=2}^{\infty} b_{m-1}x^{m-1} = 0.$$

The coefficient of $\ln x$ in (8) is 0, since y_1 is a solution of (7). From the series representation of y_1 we see that

$$2xy_1' + 2y_1 = 2\left[\sum_{m=0}^{\infty} \frac{m-1}{(m!)^2} x^{m-1} + \sum_{m=0}^{\infty} \frac{1}{(m!)^2} x^{m-1}\right]$$
$$= \sum_{m=0}^{\infty} \frac{2m}{(m!)^2} x^{m-1} = \sum_{m=1}^{\infty} \frac{2}{(m-1)!m!} x^{m-1}.$$

After substituting this into (8) and combining terms with like powers, we arrive at the equation

$$(2 + b_1) + \sum_{m=2}^{\infty} \left(\frac{2}{(m-1)!m!} + m^2 b_m - b_{m-1}\right) x^{m-1} = 0.$$

Thus $b_1 = -2$, and the recurrence relation for the b_m's is

$$b_m = \frac{1}{m^2}\left(b_{m-1} - \frac{2}{(m-1)!m!}\right).$$

From this relation we can calculate as many coefficients as we wish, although it would be difficult to find a closed formula for b_m in terms of m. Using the first five coefficients, we have

$$y_2 = \ln x \sum_{m=0}^{\infty} \frac{1}{(m!)^2} x^{m-1} - \left(2 + \frac{3}{4}x + \frac{11}{108}x^2 + \frac{25}{3,456}x^3 + \frac{137}{432,000}x^4 + \cdots\right).$$

Hence the general solution of (7) is

$$y = c_1 y_1 + c_2 y_2$$
$$= (c_1 + c_2 \ln x) \sum_{m=0}^{\infty} \frac{1}{(m!)^2} x^{m-1}$$
$$- c_2 \left(2 + \frac{3}{4}x + \frac{11}{108}x^2 + \frac{25}{3,456}x^3 + \frac{137}{432,000}x^4 + \cdots\right),$$

where c_1 and c_2 are arbitrary constants.   ∎

EXAMPLE 4   Case $r_1 - r_2$ is an integer

Solve the differential equation

(9) $$x^2 y'' - xy' + (x - 3)y = 0, \quad x > 0.$$

Solution We have $p_0 = -1$, $q_0 = -3$, and so the indicial equation is $r^2 - 2r - 3 = 0$. The indicial roots are $r_1 = 3$ and $r_2 = -1$. We are in Case 3. For a first solution we try $y_1 = \sum_{m=0}^{\infty} a_m x^{m+3}$. Substituting in (9) and solving for the coefficients a_m, we arrive at the recurrence relation

$$a_m = \frac{-1}{m^2 + 4m} a_{m-1},$$

from which we get

(10) $$y_1 = a_0 x^3 \left(1 - \frac{1}{5} x + \frac{1}{60} x^2 - \frac{1}{1,260} x^3 + \frac{1}{40,320} x^4 - \cdots \right).$$

The details are left as an exercise (Exercise 31). The second solution is of the form

$$y = ky_1 \ln x + \sum_{m=0}^{\infty} b_m x^{m-1},$$

where k may or may not be 0. We substitute this in (9), simplify, use the fact that y_1 is a solution, and get

$$\overbrace{[x^2 y_1'' - xy_1' + (x - 3)y_1]}^{= 0} \ln x + 2k(xy_1' - y_1)$$

$$+ \sum_{m=0}^{\infty} b_m(m - 1)(m - 2)x^{m-1} - \sum_{m=0}^{\infty} b_m(m - 1)x^{m-1}$$

$$- \sum_{m=0}^{\infty} 3b_m x^{m-1} + \sum_{m=1}^{\infty} b_{m-1}x^{m-1} = 0.$$

The lowest power of x appearing in the term $2k(xy_1' - y_1)$ is x^3, so we obtain the following equation upon simplification:

(11) $$(b_0 - 3b_1) + (b_1 - 4b_2)x + (b_2 - 3b_3)x^2 + 2k(xy_1' - y_1)$$

$$+ \sum_{m=4}^{\infty} [b_m(m^2 - 4m) + b_{m-1}]x^{m-1} = 0.$$

Taking $b_0 = 1$, setting the first three coefficients equal to 0, and solving, we obtain

$$b_1 = \frac{1}{3}, \quad b_2 = \frac{1}{12}, \quad b_3 = \frac{1}{36}.$$

Now from (10) it follows that

$$2k(xy_1' - y_1) = 2k \left(2x^3 - \frac{3}{5} x^4 + \frac{1}{15} x^5 - \frac{1}{252} x^6 + \cdots \right),$$

and so, setting the coefficient of x^3 in (11) equal to 0, we get

$$4k + b_3 = 0.$$

Thus $k = -\frac{1}{4}b_3 = -\frac{1}{144}$. We can now continue to use (11) to find as many of the coefficients b_4, b_5, ... as we wish, by setting the coefficients of x^4, x^5, ... in turn equal to zero. Omitting the details, we find that b_4 is arbitrary (so we choose $b_4 = 1$), and so $b_5 = -\frac{121}{600}$. Hence

$$y_2 = -\frac{1}{144}\, y_1 \ln x + x^{-1}\left(1 + \frac{1}{3}\, x + \frac{1}{12}x^2 + \frac{1}{36}\, x^3 + x^4 - \frac{121}{600}\, x^5 + \cdots\right).$$

So the general solution of (9) is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= x^3\left(c_1 - c_2\,\frac{\ln x}{144}\right)\left(1 - \frac{1}{5}\, x + \frac{1}{60}\, x^2 - \frac{1}{1,260}\, x^3 + \frac{1}{4,032}\, x^4 - \cdots\right) \\
&\quad + c_2 x^{-1}\left(1 + \frac{1}{3}\, x + \frac{1}{12}\, x^2 + \frac{1}{36}\, x^3 + x^4 - \frac{121}{600}\, x^5 + \cdots\right),
\end{aligned}
$$

where c_1 and c_2 are arbitrary constants. ∎

In the preceding example the second solution y_2 contained a logarithmic term, since the constant k turned out to be nonzero. On the other hand, Example 1 deals with a Case 3 equation in which this does not happen. Indeed, the indicial equation is $r(r-1) + 2r = 0$ with roots $r_1 = 0$ and $r_2 = -1$ that differ by a positive integer, and the solutions of the equation, $y_1 = \frac{\sin x}{x}$ and $y_2 = \frac{\cos x}{x}$, do not contain a logarithmic term.

Use of the Reduction of Order Formula

In all three cases of the Frobenius method, once we have found y_1 (using Theorem 1), a second solution can be derived by using the reduction of order formula:

$$(12) \qquad\qquad y_2(x) = y_1(x) \int \frac{e^{-\int p(x)\,dx}}{y_1^2(x)}\, dx.$$

To get the series for y_2, we expand the integrand and integrate term by term. This process is illustrated by our next example and is justified in the appendix of this section. The task of expanding is greatly simplified with the help of a computer.

EXAMPLE 5   The reduction of order formula

Find the general solution of the differential equation

$$x^2 y'' + xy' + (x^2 - 1)y = 0, \quad x > 0.$$

Solution The indicial equation is $r^2 - 1 = 0$ with indicial roots $r_1 = 1$ and $r_2 = -1$. (Note that the indicial roots differ by an integer, so we cannot assume at the outset

that the solutions are both of the form (3).) By Theorem 1, one solution is of the form $y_1 = x^{r_1} \sum_{m=0}^{\infty} a_m x^m$ with $r_1 = 1$. Plugging in the differential equation and solving for the coefficients as in the previous examples, we find

$$y_1 = x \left(1 - \frac{x^2}{8} + \frac{x^4}{192} - \frac{x^6}{9,216} + \frac{x^8}{737,280} - \frac{x^{10}}{88,473,600} + \cdots \right)$$

(Exercise 32). We find a second linearly independent solution with the help of the reduction of order formula. Putting y_1 in (12) we get

$$
\begin{aligned}
y_2(x) &= y_1 \int \frac{e^{-\int 1/x\,dx}}{y_1^2}\,dx \qquad \left(e^{-\int 1/x\,dx} = \frac{1}{x} \right) \\[2mm]
&= y_1 \int \frac{1}{x^3 \left(1 - \frac{x^2}{8} + \frac{x^4}{192} - \frac{x^6}{9,216} + \frac{x^8}{737,280} - \frac{x^{10}}{88,473,600} + \cdots \right)^2}\,dx \\[2mm]
&= y_1 \int \frac{1}{x^3 \left(1 - \frac{x^2}{4} + \frac{5\,x^4}{192} - \frac{7\,x^6}{4,608} + \frac{7\,x^8}{122,880} - \frac{11\,x^{10}}{7,372,800} + \cdots \right)}\,dx \\
&\qquad \text{(Squaring)} \\[2mm]
&= y_1 \int \left(\frac{1}{x^3} + \frac{1}{4x} + \frac{7\,x}{192} + \frac{19\,x^3}{4.608} + \frac{149\,x^5}{368,640} + \frac{803\,x^7}{22,118,400} + \cdots \right)\,dx \\
&\qquad \text{(Series expansion or long division)} \\[2mm]
&= y_1 \left(\frac{-1}{2\,x^2} + \frac{1}{4}\ln x + \frac{7x^2}{384} + \frac{19\,x^4}{18,432} + \frac{149\,x^6}{2,211,840} + \frac{803\,x^8}{176.947,200} + \cdots \right) \\
&\qquad \text{(Integrate term by term)} \\[2mm]
&= y_1 \frac{\ln x}{4} + y_1 \left(\frac{-1}{2x^2} + \frac{7x^2}{384} + \frac{19x^4}{18,432} + \frac{149x^6}{2,211,840} + \frac{803x^8}{176,947,200} + \cdots \right).
\end{aligned}
$$

This determines a second linearly independent solution. ∎

Example 5 shows that the reduction of order formula is a handy tool for checking for the presence of a logarithmic term in the solution.

Appendix: Further Proofs Related to the Frobenius Method

In this appendix we use the reduction of order formula to derive the second solution in the Frobenius problems. given the first one by Theorem 1.   We restrict our attention to the case $x > 0$.

Recall that $xp(x)$ has a power series at $x = 0$. Thus we have

$$
\begin{aligned}
xp(x) &= p_0 + p_1 x + p_2 x^2 + \cdots, \\
p(x) &= \frac{p_0}{x} + p_1 + p_2 x + \cdots.
\end{aligned}
$$

We denote the function $p_1 + p_2 x + \cdots$ by $a(x)$ (note that $a(x)$ has a power series expansion at $x = 0$), and so

$$p(x) = \frac{p_0}{x} + a(x).$$

Integrating and taking the exponential, we get

(13) $$e^{-\int p(x)\,dx} = x^{-p_0} G(x),$$

where G is the function $e^{-\int a(x)\,dx}$, and, in particular, $G(0) \neq 0$. (Note that G has a power series expansion at $x = 0$.) Now, using Theorem 1, we can write

$$(14) \qquad\qquad \frac{1}{y_1^2} = \frac{1}{x^{2r_1}}\, H(x),$$

where $H(x) = 1\Big/ \left(\sum_{m=0}^{\infty} a_m x^m\right)^2$. Since $a_0 \neq 0$, the function $H(x)$ has a power series expansion at 0 and $H(0) \neq 0$. Combining (12), (13), and (14), we find that

$$(15) \qquad\qquad y_2 = y_1 \int x^{-p_0 - 2r_1} K(x)\, dx,$$

where $K(x)$ has a power series expansion about 0, and $K(0) \neq 0$. We write

$$K(x) = k_0 + k_1 x + k_2 x^2 + \cdots, \qquad k_0 \neq 0.$$

At this point, we establish a connection between $r_1,\, r_2,\, p_0$. Since r_1 and r_2 are the roots of equation (5), we have $r_1 + r_2 = 1 - p_0$. Thus (15) becomes

$$(16) \qquad y_2 = y_1 \int x^{-(r_1 - r_2 + 1)} (k_0 + k_1 x + k_2 x^2 + \cdots)\, dx, \quad k_0 \neq 0.$$

Note that a logarithmic term in y_2 will appear if and only if a term in x^{-1} occurs in the integrand (after multiplying through by $x^{-(r_1 - r_2 + 1)}$). This is why $r_1 - r_2$ determines the nature of the second solution.

Case 1: If $r_1 - r_2$ is not an integer, clearly there can be no term in x^{-1}, and hence no logarithmic term in y_2.

Case 2: If $r_1 = r_2$, then the very first term yields the logarithmic term in y_2, since $k_0 \neq 0$.

Case 3: If $r_1 - r_2$ is a positive integer, say $r_1 - r_2 = n$, we may or may not have a logarithmic term, depending on whether k_n is 0.

Exercises A.5

In Exercises 1-6, decide whether $x = 0$ is an ordinary point or a singular point. In case it is a singular point, determine if it is a regular singular point.

1. $y'' + (1 - x^2)y' + xy = 0$.
2. $xy'' + \sin x\, y' + \frac{1}{x} y = 0$.

3. $x^3 y'' + x^2 y' + y = 0$.
4. $\sin x\, y'' + y' + \frac{1}{x} y = 0$.

5. $x^2 y'' + (1 - e^x)y' + xy = 0$.
6. $3\, xy'' + 2 y' - \frac{1}{3x} y = 0$.

In Exercises 7-22, (a) check that $x = 0$ is a regular singular point. (b) Determine which case of the Frobenius method applies. (c) Determine at least three nonzero terms in each of two linearly independent series solutions. Take $x > 0$.

7. $4\, xy'' + 6 y' + y = 0$.
8. $4\, xy'' + 6 y' - y = 0$.

9. $4 x^2\, y'' - 14\, xy' + (20 - x)\, y = 0$.
10. $4x\, y'' + 2 y' + y = 0$.

11. $2 x\, y'' + (1 + x)\, y' + y = 0$.
12. $y'' - \frac{1}{2x} y' + \frac{1}{x} y = 0$.

13. $x\, y'' + (1 - x)\, y' + y = 0$.
14. $x\, y'' + 2 y' - x y = 0$.

15. $x\, y'' + 2\,(1 + x)\, y' + (x + 2)\, y = 0$.
16. $x^2\, y'' + x y' + (x^2 - \frac{1}{4})y = 0$.

17. $x^2 y'' + 4 x y' + (2 - x^2) y = 0.$

18. $x y'' + (x + 2) y' + (x + 1) y = 0.$

19. $x y'' + 3 y' + \frac{1+x}{x} y = 0.$

20. $x^2 y'' + 4 x y' + (2 + x^2) y = 0.$

21. $x y'' + y' - \frac{1+x}{x} y = 0.$

22. $4 x y'' + 2 y' + x y = 0.$

In Exercises 23–28, determine which case of the Frobenius method applies. and then solve the equation using Theorem 1 and reduction of order as we did in Example 5. Does your second solution contain a logarithmic term?

23. $4 x^2 y'' - 14 x y' + (20 + x) y = 0, \quad x > 0.$

24. $y'' - \frac{1+x^2}{2x} y' + \frac{1}{2x^3} y = 0, \quad x > 0.$

25. $y'' + \frac{1}{x} y' + \frac{x-1}{x^3} y = 0, \quad x > 0.$

26. $y'' - \frac{1+x}{2x} y' + \frac{1+2x}{2x^2} y = 0, x > 0.$

27. $x(1 - x) y'' + (1 - 3x) y' - y = 0, \quad 0 < x < 1.$

28. $x(1 - x) y'' - \frac{1}{4} y = 0, \quad 0 < x < 1.$

29. (a) Make the change of variables $u = xy$ in the differential equation of Example 1 and show that it becomes $u'' + u = 0.$
(b) Use (a) to derive the solutions $y_1 = \frac{\sin x}{x}$ and $y_2 = \frac{\cos x}{x}.$

30. Derive the first solution in Example 3.

31. Supply the details leading to the first solution in Example 4.

32. Supply the details leading to the first solution in Example 5.

Euler's equation can be used to illustrate the solutions in all three cases of the Frobenius method. In Exercises 33–36, (a) show that $x = 0$ is a regular singular point, and determine which case of the Frobenius method applies.   (b)   Find two linearly independent solutions. using the methods of Section A.3.

33. $y'' + \frac{1}{x} y' - \frac{2}{x^2} y = 0.$

34. $y'' + \frac{2}{x} y' - \frac{3}{x^2} y = 0.$

35. $y'' + \frac{1}{x} y' + \frac{1}{9x^2} y = 0.$

36. $y'' + \frac{1}{2x} y' + \frac{1}{2x^2} y = 0.$

37. (a) Show that $x = 0$ is a regular singular point of the differential equation

$$(\sin x)^2 y'' + \tan x \, y' + (\cos x)^2 y = 0.$$

(b) Find the indicial roots and determine which case of the Frobenius method applies.
(c) Verify that $y_1 = \sin(\ln(\sin x))$ and $y_2 = \cos(\ln(\sin x))$ are solutions of the differential equation for $0 < x < R$, where R is a positive number.   How large can R be?
(d) Make a change of variables $u = \sin x$ and transform the given equation to an Euler equation.   Then derive the solutions given in (c).

(e) Plot the graphs of the solutions.   Explain the behavior of the graphs near 0 and π.

38. Consider the equation

$$x\,y'' - (2 + x)\,y' + 2\,y = 0, \quad x > 0.$$

(a) Show that $x = 0$ is a regular singular point. Find the indicial equation and the indicial roots, and conclude that we are in Case 3 of the Frobenius method.

(b) Use the method of Frobenius to find a Frobenius series solution corresponding to the larger root r_1.

(c) Identify the solution in (b) as $3!(e^x - 1 - x - \frac{x^2}{2})$.

(d) Even though the roots differ by an integer, there exists a second Frobenius series solution. Using the method of Frobenius, show that e^x or $1 + x + x^2/2$ can be taken as a second solution. [Hint: Argue that the coefficient b_3 is arbitrary.]

39. Consider the equation

$$xy'' - (n + x)\,y' + n\,y = 0, \quad x > 0,$$

where n is a nonnegative integer.

(a) Show that $x = 0$ is a regular singular point. Find the indicial equation and the indicial roots, and conclude that we are in Case 3 of the Frobenius method.

(b) Use the method of Frobenius to find a Frobenius series solution corresponding to the larger root r_1.

(c) Identify the solution in (b) as

$$(n + 1)!(e^x - 1 - x - \frac{x^2}{2} - \cdots - \frac{x^n}{n!}).$$

(d) Using the method of Frobenius, show that e^x or $1 + x + x^2/2 + \cdots + \frac{x^n}{n!}$ can be taken as a second solution. [Hint: Argue that the coefficient b_{n+1} is arbitrary.]