# Chapter 2

# **1-Dimensional Waves**

In this Chapter we first consider first order PDE and then move to 1dimensional wave equation which we analyze by the *method of characteristics*.

### 2.1 First order PDEs

### 2.1.1 Introduction

Consider PDE

$$au_t + bu_x = 0. (2.1.1)$$

Note that the left-hand expression is a *derivative of u along vector field*  $\ell = (a, b)$ . Consider an *integral lines* of this vector field:

$$\frac{dt}{a} = \frac{dx}{b}.\tag{2.1.2}$$

*Remark* 2.1.1. (a) Recall from ODE course that an *integral line* of the vector field is a line, tangent to it in each point.

(b) Often it is called *directional derivative* but also often then  $\ell$  is *normalized*, replaced by the unit vector of the same direction  $\ell^0 = \ell/|\ell|$ .

### 2.1.2 Constant coefficients

If a and b are constant then integral curves are just straight lines t/a - x/b = C where C is a constant along integral curves and it labels them (at least as

long as we consider the whole plane (x, t)). Therefore u depends only on C:

$$u = \phi\left(\frac{t}{a} - \frac{x}{b}\right) \tag{2.1.3}$$

where  $\phi$  is an arbitrary function.

This is a *general solution* of our equation.

Consider initial value condition  $u|_{t=0} = f(x)$ . It allows us define  $\phi$ :  $\phi(-x/b) = f(x) \implies \phi(x) = f(-bx)$ . Plugging in u we get

$$u = f(x - ct) \qquad \text{with } c = b/a. \tag{2.1.4}$$

It is a solution of IVP

$$\begin{cases} au_t + bu_x = 0, \\ u(x,0) = f(x). \end{cases}$$
(2.1.5)

Obviously we need to assume that  $a \neq 0$ .

Also we can rewrite general solution in the form u(x,t) = f(x-ct) where now f(x) is another arbitrary function.

**Definition 2.1.1.** Solutions  $u = \chi(x - ct)$  are running waves where c is a propagation speed.

#### visual examples

### 2.1.3 Variable coefficients

If a and/or b are not constant these integral lines are curves.

Example 2.1.1. Consider equation  $u_t + tu_x = 0$ . Then equation of the integral curve is  $\frac{dt}{1} = \frac{dx}{t}$  or equivalently tdt - dx = 0 which solves as  $x - \frac{1}{2}t^2 = C$  and therefore  $u = \phi(x - \frac{1}{2}t^2)$  is a general solution to this equation.

One can see easily that  $u = f(x - \frac{1}{2}t^2)$  is a solution of IVP.

Example 2.1.2. (a) Consider the same equation  $u_t + tu_x = 0$  but let us consider IVP as x = 0: u(0,t) = g(t). However it is not a good problem: first, some integral curves intersect line x = 0 more than once and if in different points of intersection of the same curve initial values are different we get a contradiction (therefore problem is not solvable for g which are not even functions).



(b) On the other hand, if we consider even function g (or equivalently impose initial condition only for t > 0) then u is not defined on the curves which are not intersecting x = 0 (which means that u is not defined for  $x > \frac{1}{2}t^2$ .)

In Part (a) of this example both *solvability* and *uniqueness* are broken; in Part (b) only *uniqueness* is broken. But each integral line intersects  $\{(x,t): t = 0\}$  exactly once, so IVP of Example 2.1.1 is *well-posed*.

### 2.1.4 Right-hand expression

Consider the same equation albeit with the right-hand expression

$$au_t + bu_x = f, \qquad f = f(x, t, u).$$
 (2.1.6)

Then as  $\frac{dt}{a} = \frac{dx}{b}$  we have  $du = u_t dt + u_x dx = (au_t + bu_x)\frac{dt}{a} = f\frac{dt}{a}$  and therefore we expand our ordinary equation (2.1.2) to

$$\frac{dt}{a} = \frac{dx}{b} = \frac{du}{f}.$$
(2.1.7)

*Example 2.1.3.* Consider problem  $u_t + u_x = x$ . Then  $\frac{dx}{1} = \frac{dt}{1} = \frac{du}{x}$ . Then x - t = C and  $u - \frac{1}{2}x^2 = D$  and we get  $u - \frac{1}{2}x^2 = \phi(x - t)$  as relation between C and D both of which are constants along integral curves. Here  $\phi$  is an arbitrary function. So  $u = \frac{1}{2}x^2 + \phi(x-t)$  is a general solution.

Imposing initial condition  $u|_{t=0} = \tilde{0}$  (sure, we could impose another condition) we have  $\phi(x) = -\frac{1}{2}x^2$  and plugging into u we get

$$u(x,t) = \frac{1}{2}x^2 - \frac{1}{2}(x-t)^2 = xt - \frac{1}{2}t^2.$$

*Example 2.1.4.* Consider  $u_t + xu_x = xt$ . Then  $\frac{dt}{1} = \frac{dx}{x} = \frac{du}{xt}$ . Solving the first equation  $t - \ln(x) = -\ln(C) \implies x = Ce^t$  we get integral curves.

Now we have

$$\frac{du}{xt} = dt \implies du = xtdt = Cte^t dt$$
$$\implies u = C(t-1)e^t + D = x(t-1) + D$$

where D must be constant along integral curves and therefore  $D = \phi(xe^{-t})$ with an arbitrary function  $\phi$ . So  $u = x(t-1) + \phi(xe^{-t})$  is a general solution of this equation.

Imposing initial condition  $u|_{t=0} = 0$  (sure, we could impose another condition) we have  $\phi(x) = x$  and then  $u = x(t - 1 + e^{-t})$ .

#### 2.1.5Linear and semilinear equations

**Definition 2.1.2.** (a) If a = a(x, t) and b = b(x, t) equation is semilinear. In this case we first define integral curves which do not depend on u and then find u as a solution of ODE along these curves.

(b) Furthermore if f is a linear function of u: f = c(x, t)u + q(x, t) original equation is *linear*. In this case the last ODE is also linear.

*Example 2.1.5.* Consider  $u_t + xu_x = u$ . Then  $\frac{dt}{1} = \frac{dx}{x} = \frac{du}{u}$ . Solving the first equation  $t - \ln(x) = -\ln(C) \implies x = Ce^t$  we get integral curves. Now we have

$$\frac{du}{u} = dt \implies \ln(u) = t + \ln(D) \implies u = De^t = \phi(xe^{-t})e^t$$

which is a general solution of this equation.

Imposing initial condition  $u|_{t=0} = x^2$  (sure, we could impose another condition) we have  $\phi(x) = x^2$  and then  $u = x^2 e^{-t}$ .

*Example 2.1.6.* Consider  $u_t + xu_x = -u^2$ . Then  $\frac{dt}{1} = \frac{dx}{x} = -\frac{du}{u^2}$ . Solving the first equation we get integral curves  $x = Ce^t$ . Now we have

$$-\frac{du}{u^2} = dt \implies u^{-1} = t + D \implies u = (t + \phi(xe^{-t}))^{-1}.$$

which is a general solution of this equation.

Imposing initial condition  $u|_{t=0} = -1$  we get  $\phi = -1$  and then  $u = (t-1)^{-1}$ . This solution "blows up" as t = 1 (no surprise, we got it, solving *non-linear* ODE.

### 2.1.6 Quasilinear equations

**Definition 2.1.3.** If a and/or b depend on u this is quasininear equation.

For such equations integral curves depend on the solution which can lead to breaking of solution. Indeed, while equations (2.1.7)

$$\frac{dt}{a} = \frac{dx}{b} = \frac{du}{f}.$$

define curves in 3-dimensional space of (t, x, u), which do not intersect, their projections on (t, x)-plane can intersect and then u becomes a multivalued function, which is not allowed.

*Example* 2.1.7. Consider Burgers equation  $u_t + uu_x = 0$  (which is an extremely simplified model of gas dynamics). We have  $\frac{dt}{1} = \frac{dx}{u} = \frac{du}{0}$  and therefore u = const along integral curves and therefore integral curves are x - ut = C.

Consider initial problem u(x, 0) = g(x). We take initial point (y, 0), find here u = g(y), then x - g(y)t = y (because x = y + ut and u is constant along integral curves) and we get u = g(y) where y = y(x, t) is a solution of equation x = g(y)t + y.

The trouble is that by implicit function theorem we can define y for all x only if  $\frac{\partial}{\partial y}(y+g(y)t)$  does not vanish. So,

$$g'(y)t + 1 \neq 0. \tag{2.1.8}$$

This is possible for all t > 0 if and only if  $g'(y) \ge 0$  i.e. f is a monotone non-decreasing function.



Figure 2.1: Integral lines for Burgers equation if  $f(x) = \tanh(x)$ .

Equivalently, u is defined as an implicit function by

$$u = g(x - ut).$$
 (2.1.9)

Then the implicit function theorem could be applied iff

$$\frac{\partial}{\partial u} (u - g(x - ut)) = 1 + g'(x - ut)t \neq 0,$$

which leads us to the previous conclusion.

So, *classical solution* breaks for some t > 0 if g is not a monotone non-decreasing function. A proper understanding of the *global solution* for such equation goes well beyond our course. Some insight is provided by the analysis in Section 12.1

Example 2.1.8. Traffic flow is considered in Appendix 2.1.A.

### 2.1.7 IBVP

Consider IBVP (initial-boundary value problem) for constant coefficient equation

$$\begin{cases} u_t + cu_x = 0, & x > 0, \ t > 0, \\ u_{t=0} = f(x) & x > 0. \end{cases}$$
(2.1.10)

The general solution is  $u = \phi(x - ct)$  and plugging into initial data we get  $\phi(x) = f(x)$  (as x > 0).



Figure 2.2: Integral lines for Burgers equation if  $f(x) = -\tanh(x)$ .

So, u(x,t) = f(x - ct). Done!—Not so fast: f is defined only for x > 0so u is defined for x - ct > 0 (or x > ct). It covers the whole quadrant if  $c \le 0$  (so waves run to the left) and only in this case we are done.

On the other hand, if c > 0 (waves run to the right) u is not defined as x < ct and to define it here we need a *boundary condition* at x = 0.

So we get IBVP (initial-boundary value problem)

$$\begin{cases} u_t + cu_x = 0, & x > 0, t > 0, \\ u_{t=0} = f(x) & x > 0, \\ u_{x=0} = g(t) & t > 0. \end{cases}$$
(2.1.11)

Then we get  $\phi(-ct) = g(t)$  as t > 0 which implies  $\phi(x) = g(-\frac{1}{c}x)$  as x < 0 and then  $u(x,t) = g(-\frac{1}{c}(x-ct)) = g(t-\frac{1}{c}x)$  as x < ct.

So solution is

$$u = \begin{cases} f(x - ct) & x > ct, \\ g(t - \frac{1}{c}x) & x < ct. \end{cases}$$
(2.1.12)

Remark 2.1.2. Unless f(0) = g(0) this solution is discontinuous as x = ct. Therefore it is not a classical solution-derivatives do not exist. However it is still a *weak solution* (see Section 11.4) and *solution in the sense of distributions* (see Section 11.1).



### Problems to Section 2.1

Problem 1. (a) Draw characteristics and find the general solution to each of the following equations

 $\begin{aligned} &2u_t + 3u_x = 0; & u_t + tu_x = 0; \\ &u_t - tu_x = 0; & u_t + t^2u_x = 0; \\ &u_t + xu_x = 0; & u_t + txu_x = 0; \\ &u_t + x^2u_x = 0; & u_t + (x^2 + 1)u_x = 0; \\ &u_t + (t^2 + 1)u_x = 0; & (x + 1)u_t + u_x = 0; \\ &(x + 1)^2u_t + u_x = 0; & (x^2 + 1)u_t + u_x = 0; \\ &(x^2 - 1)u_t + u_x = 0. \end{aligned}$ 

(b) Consider IVP problem  $u|_{t=0} = f(x)$  as  $-\infty < x < \infty$ ; does solution always exists? If not, what conditions should satisfy f(x)? Consider separately t > 0 and t < 0.

(c) Where this solution is uniquely determined? Consider separately t > 0 and t < 0.

(d) Consider this equation in  $\{t > 0, x > 0\}$  with the initial condition  $u|_{t=0} = f(x)$  as x > 0; where this solution defined? Is it defined everywhere

in  $\{t > 0, x > 0\}$  or do we need to impose condition at x = 0? In the latter case impose condition  $u|_{x=0} = g(t)$  (t > 0) and solve this IVBP;

(e) Consider this equation in  $\{t > 0, x < 0\}$  with the initial condition  $u|_{t=0} = f(x)$  as x < 0; where this solution defined? Is it defined everywhere in  $\{t > 0, x < 0\}$  or do we need to impose condition at x = 0? In the latter case impose condition  $u|_{x=0} = g(t)$  (t > 0) and solve this IVBP;

(f) Consider problems (d) as t < 0;

(g) Consider problems (e) as t < 0;

Problem 2. (a) Find the general solution to each of the following equations

$$xu_x + yu_y = 0,$$
  
$$xu_x - yu_y = 0$$

in  $\{(x, y) \neq (0, 0)\}$ ; when this solution is continuous at (0, 0)? Explain the difference between these two cases;

(b) Find the general solution to each of the following equations

$$yu_x + xu_y = 0,$$
  
$$yu_x - xu_y = 0$$

in  $\{(x, y) \neq (0, 0)\}$ ; when this solution is continuous at (0, 0)? Explain the difference between these two cases;

*Problem* 3. In the same way consider equations

$$(x^{2} + 1)yu_{x} + (y^{2} + 1)xu_{y} = 0;$$
  
(x<sup>2</sup> + 1)yu<sub>x</sub> - (y<sup>2</sup> + 1)xu<sub>y</sub> = 0.

Problem 4. Find the solutions of

$$\begin{cases} u_x + 3u_y = xy, \\ u|_{x=0} = 0; \end{cases} \qquad \qquad \begin{cases} u_x + 3u_y = u, \\ u|_{x=0} = y. \end{cases}$$

*Problem* 5. Find the general solutions to each of

$$yu_x - xu_y = x; \qquad yu_x - xu_y = x^2;$$
  
$$yu_x + xu_y = x; \qquad yu_x + xu_y = x^2.$$

In one instance solution does not exist. Explain why.

Problem 2. (a) (a)

$$u_t + 3u_x - 2u_y = x; \qquad u_t + xu_x + yu_y = x;$$
$$u_t + xu_x - yu_y = x; \qquad u_t + yu_x + xu_y = x;$$
$$u_t + yu_x - xu_y = x.$$

(b) Solve IVP u(x, y, 0) = 0.

Problem 3. (a) Find the general solution to each of the following equations

$u_t + 3u_x - 2u_y = u;$	$u_t + xu_x + yu_y = u;$
$u_t + xu_x - yu_y = u;$	$u_t + yu_x + xu_y = u;$
$u_t + yu_x - xu_y = u;$	$u_t + 3u_x - 2u_y = xyu.$

(b) Solve IVP u(x, y, 0) = f(x, y).

### 2.3 Homogeneous 1D wave equation

### 2.3.1 Physical examples

Consider equation

$$u_{tt} - c^2 u_{xx} = 0. (2.3.1)$$

*Remark* 2.3.1. (a) As we mentioned in Subsection 1.4.1 this equation describes a lot of things.

(b) c has a dimension of the speed. In the example above c is a *speed of sound*.

*Example 2.3.1.* (a) This equation describes oscillations of the string Example 1.4.1.

(b) It also describes 1-dimensional gas oscillations (see Example 1.4.3).

(c) Further, this equation with  $c = c_{\parallel}$  also describes *compression-rarefication* waves in the elastic 1-dimensional media. Then u(x, t) is displacement along x.

(d) And also this equation with  $c = c_{\perp} < c_{\parallel}$  also describes *sheer waves* in the elastic 1-dimensional media. Then u(x,t) is displacement in the direction, perpendicular to x.

### 2.3.2 General solution

Let us rewrite formally equation (2.3.1) as

$$(\partial_t^2 - c^2 \partial_x^2) u = (\partial_t - c \partial_x) (\partial_t + c \partial_x) u = 0.$$
(2.3.2)

Denoting  $v = (\partial_t + c\partial_x)u = u_t + cu_x$  and  $w = (\partial_t - c\partial_x)u = u_t - cu_x$  we have

$$v_t - cv_x = 0, (2.3.3)$$

$$w_t + cw_x = 0. (2.3.4)$$

But from Section 2.1 we know how to solve these equations

$$v = 2c\phi'(x+ct),$$
 (2.3.5)

$$w = -2c\psi'(x - ct) \tag{2.3.6}$$

where  $\phi'$  and  $\psi'$  are arbitrary functions. We find convenient to have factors 2c and -2c and to denote by  $\phi$  and  $\psi$  their primitives (a.k.a. indefinite integrals).

Recalling definitions of v and w we have

$$u_t + cu_x = 2c\phi'(x + ct),$$
  

$$u_t - cu_x = -2c\psi'(x - ct).$$

Observe that the right-hand side of (2.3.5) equals to  $(\partial_t + c\partial_x)\phi(x + ct)$ and therefore  $(\partial_t + c\partial_x)(u - \phi(x + ct)) = 0$ . Then  $u - \phi(x + ct)$  must be a function of x - ct:  $u - \phi(x + ct) = \chi(x - ct)$  and plugging into (2.3.6) we conclude that  $\chi = \psi$  (up to a constant, but both  $\phi$  and  $\psi$  are defined up to some constants).

Therefore

$$u = \phi(x + ct) + \psi(x - ct)$$
 (2.3.7)

is a general solution to (2.3.1). This solution is a superposition of two waves  $u_1 = \phi(x + ct)$  and  $u_2 = \psi(x - ct)$  running to the left and to the right respectively with the speed c. So c is a propagation speed.

*Remark* 2.3.2. Adding constant C to  $\phi$  and -C to  $\psi$  we get the same solution u. However it is the only arbitrariness.

Visual examples (animation)

### 2.3.3 Cauchy problem

Let us consider IVP (*initial-value problem*, aka Cauchy problem) for (2.3.1):

$$u_{tt} - c^2 u_{xx} = 0, (2.3.8)$$

$$u|_{t=0} = g(x),$$
  $u_t|_{t=0} = h(x).$  (2.3.9)

Plugging (2.3.7)  $u = \phi(x + ct) + \psi(x - ct)$  into initial conditions we have

$$\phi(x) + \psi(x) = g(x), \tag{2.3.10}$$

$$c\phi'(x) - c\psi'(x) = h(x) \implies \phi(x) - \psi(x) = \frac{1}{c} \int^x h(y) \, dy.$$
 (2.3.11)

Then

$$\phi(x) = \frac{1}{2}g(x) + \frac{1}{2c}\int^x h(y)\,dy,\qquad(2.3.12)$$

$$\psi(x) = \frac{1}{2}g(x) - \frac{1}{2c}\int^x h(y)\,dy. \tag{2.3.13}$$

Plugging into (2.3.7)

$$u(x,t) = \frac{1}{2}g(x+ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} h(y)\,dy + \frac{1}{2}g(x-ct) - \frac{1}{2c}\int_{x-ct}^{x-ct} h(y)\,dy$$

we get D'Alembert formula

$$u(x,t) = \frac{1}{2} \left[ g(x+ct) + g(x-ct) \right] + \frac{1}{2c} \int_{x-ct}^{x+ct} h(y) \, dy.$$
(2.3.14)

*Remark* 2.3.3. Later we generalize it to the case of inhomogeneous equation (with the right-hand expression f(x, t) in (2.3.8)).

#### Problems to Section 2.3

Problem 1. Find the general solutions of

$$u_{tt} - u_{xx} = 0; u_{tt} - 4u_{xx} = 0; u_{tt} - 9u_{xx} = 0; 4u_{tt} - u_{xx} = 0; 4u_{tt} - u_{xx} = 0; u_{tt} - 9u_{xx} = 0.$$

Problem 2. Solve IVP

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, \\ u_{t=0} = g(x), \quad u_t|_{t=0} = h(x) \end{cases}$$
(2..1)

with

$$g(x) = \begin{cases} 0 & x < 0, \\ 1 & x \ge 0, \end{cases} \qquad h(x) = 0;$$
$$g(x) = \begin{cases} 1 & |x| < 1, \\ 0 & |x| \ge 1, \end{cases} \qquad h(x) = 0;$$

$$g(x) = \begin{cases} 1 - |x| & |x| < 1, \\ 0 & |x| \ge 1, \end{cases} \qquad h(x) = 0;$$

$$g(x) = \begin{cases} 1 - x^2 & |x| < 1, \\ 0 & |x| \ge 0, \end{cases} \qquad h(x) = 0;$$

$$g(x) = \begin{cases} \cos(x) & |x| < \pi/2, \\ 0 & |x| \ge \pi/2, \end{cases} \qquad h(x) = 0;$$

$$g(x) = \begin{cases} \cos^2(x) & |x| < \pi/2, \\ 0 & |x| \ge \pi/2, \end{cases} \qquad h(x) = 0;$$

$$g(x) = \begin{cases} \sin(x) & |x| < \pi, \\ 0 & |x| \ge \pi, \end{cases} \qquad h(x) = 0;$$

$$g(x) = \begin{cases} \sin^2(x) & |x| < \pi, \\ 0 & |x| \ge \pi, \end{cases} \qquad h(x) = 0;$$

$$g(x) = 0, \qquad h(x) = \begin{cases} 0 & x < 0, \\ 1 & x \ge 0; \end{cases}$$

$$g(x) = 0, \qquad h(x) = \begin{cases} 1 - x^2 & |x| < 1, \\ 0 & |x| \ge 0; \end{cases}$$

$$g(x) = 0, \qquad h(x) = \begin{cases} 1 & |x| < 1, \\ 0 & |x| \ge 1; \end{cases}$$

$$g(x) = 0, \qquad h(x) = \begin{cases} \cos(x) & |x| < \pi/2, \\ 0 & |x| \ge \pi/2; \end{cases}$$

$$g(x) = 0, \qquad h(x) = \begin{cases} \sin(x) & |x| < \pi, \\ 0 & |x| \ge \pi. \end{cases}$$

*Problem* 3. Find solution u = u(x,t) and describe domain, where it is uniquely defined

$$u_{tt} - u_{xx} = 0;$$
  
 $u|_{t=x^2/2} = x^3;$   
 $u_t|_{t=x^2/2} = 2x.$ 

Problem 4. (a) Prove that if u solves the problem

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & -\infty < x < \infty, \\ u_{t=0} = g(x), \\ u_t|_{t=0} = 0, \end{cases}$$
(2..2)

then  $v = \int_0^t u(x, t') dt'$  solves

$$\begin{cases} v_{tt} - c^2 v_{xx} = 0 & -\infty < x < \infty, \\ v_{t=0} = 0, \\ v_t|_{t=0} = g(x). \end{cases}$$
(2..3)

(b) Also prove that if v solves (2..3) then  $u = v_t$  solves (2..2).

(c) From formula

$$u(x,t) = \frac{1}{2} (g(x+ct) + g(x-ct))$$
(2..4)

for the solution of (2..2) derive

$$v(x,t) = \frac{1}{2c} \int_{x-ct}^{x+ct} g(x') \, dx'$$
(2..5)

for the solution of (2..3).

(d) Conversely, from (2..5) for the solution of (2..3) derive (2..4) for the solution of (2..2).

Problem 5. Find solution to equation

$$Au_{tt} + 2Bu_{tx} + Cu_{xx} = 0 (2..6)$$

as

$$u = f(x - c + 1t) + g(x - c + 2t)$$
(2..7)

with arbitrary f, g and real  $c + 1 < c_2$ .

(a) What equation should satisfy  $c_1$  and  $c_2$ ?

(b) When this equation has such roots?

Problem 6. A spherical wave is a solution of the three-dimensional wave equation of the form u(r,t), where r is the distance to the origin (the spherical coordinate). The wave equation takes the form

$$u_{tt} = c^2 \left( u_{rr} + \frac{2}{r} u_r \right)$$
 (spherical wave equation). (2..8)

(a) Change variables v = ru to get the equation for v:  $v_{tt} = c^2 v_{rr}$ .

(b) Solve for v using

$$v = f(r+ct) + g(r-ct)$$
 (2..9)

and thereby solve the spherical wave equation.

(c) Use

$$v(r,t) = \frac{1}{2} \left[ \phi(r+ct) + \phi(r-ct) \right] + \frac{1}{2c} \int +r - ct^{r+ct} \psi(s) \, ds \qquad (2..10)$$

with  $\phi(r) = v(r,0), \ \psi(r) = v + t(r,0)$  to solve it with initial conditions  $u(r,0) = \Phi(r), \ u + t(r,0) = \Psi(r).$ 

(d) Find the general form of solution u to (2..8) which is continuous as r = 0.

Problem 7. Find formula for solution of the Goursat problem

$$u_{tt} - c^2 u_{xx} = 0,$$
  $x > c|t|;$  (2..11)

$$u|_{x=-ct} = g(t),$$
  $t < 0;$  (2..12)

$$u|_{x=ct} = h(t), (2..13)$$

as long as g(0) = h(0).

*Problem 8.* Find solution u=u(x,t) and describe domain, where it is uniquely defined

$$u + tt - u + xx = 0, (2..14)$$

$$|u| + t = x^2/2 = x^3,$$
  $|x| \le 1,$  (2..15)

$$|u+t| + t = x^2/2 = 2x$$
  $|x| \le 1.$  (2..16)

Explain, why we imposed restriction  $|x| \leq 1$ ?

Problem 9. Often solution in the form of travelling wave  $u = \phi(x - vt)$  is sought for more general equations. Here we are interested in the bounded solutions, especially in those with  $\phi(x)$  either tending to 0 as  $|x| \to \infty$ (solitons) or periodic (kinks). Plugging such solution to equation we get ODE for function  $\phi$ , which could be either solved or at least explored. Sure we are not interested in the trivial solution which is identically equal to 0.

(a) Find such solutions for each of the following equations

$$u + tt - c^2 u + xx + m^2 u = 0; (2..17)$$

$$u + tt - c^{2}u + xx - m^{2}u = 0; (2..18)$$

the former is Klein-Gordon equation. Describe all possible velocities v.

(b) Find such solutions for each of the following equations

$$u_t - K u_{xxx} = 0;$$
 (2..19)

$$u_t - iKu_{xx} = 0;$$
 (2..20)

$$u_{tt} + K u_{xxxx} = 0. (2..21)$$

 $Problem \ 10.$  Look for solutions in the form of travelling wave for sine-Gordon equation

$$u_{tt} - c^2 u_{xx} + \sin(u) = 0. \tag{2..22}$$

observe that resulting ODE is describing *mathematical pendulum* which could be explored. Describe all possible velocities v.

*Problem* 11. Look for solutions in the form of travelling wave for each of the following equations

$$u_{tt} - u_{xx} + u - 2u^3 = 0; (2..23)$$

$$u_{tt} - u_{xx} - u + 2u^3 = 0; (2..24)$$

(a) Describe such solutions (they are called kinks). Describe all possible velocities v.

(b) Find solitons. Describe all possible velocities v.

Problem 12. For a solution u(x,t) of the wave equation  $u_{tt} = c^2 u_{xx}$ , the energy density is defined as  $e = \frac{1}{2} (u_t^2 + c^2 u_x^2)$  and the momentum density as  $p = c u_t u_x$ .

(a) Show that

$$\frac{\partial e}{\partial t} = c \frac{\partial p}{\partial x}$$
 and  $\frac{\partial p}{\partial t} = c \frac{\partial e}{\partial x}$ . (2..25)

(b) Show that both e(x, t) and p(x, t) also satisfy the same wave equation.

Problem 13. (a) Consider wave equation  $u_{tt} - u_{xx} = 0$  in the rectangle 0 < x < a, 0 < t < b and prove that if a and b are not commensurable (i.e. a : b is not rational) then Dirichlet problem  $u|_{t=0} = u_{t=b} = u|_{x=0} = u|_{x=a} = 0$  has only trivial solution.

(b) On the other hand, prove that if a and b are commensurable then there exists a nontrivial solution  $u = \sin(px/a)\sin(qt/b)$ .

Problem 14. Generalize Problem 6: A spherical wave is a solution of the *n*-dimensional wave equation of the form u(r, t), where r is the distance to the origin (the spherical coordinate). The wave equation takes the form

$$u_{tt} = c^2 \left( u_{rr} + \frac{n-1}{r} u_r \right) \qquad \text{(spherical wave equation)} \tag{2..26}$$

(a) Show that if u satisfies (2..26), then  $r^{-1}\partial_r u(r,t)$  also satisfies (2..26) but with n replaced by n + 2.

(b) Using this and Problem 6: write down spherical wave for odd n.

(c) Describe spherical wave for n = 1.

Remark 2..4. For even n spherical waves do not exist.

## 2.4 1D-Wave equation reloaded: characteristic coordinates

### 2.4.1 Characteristic coordinates

We realize that lines x + ct = const and x - ct = const play a very special role in our analysis. We call these lines *characteristics*. Let us introduce *characteristic coordinates* 

$$\begin{cases} \xi = x + ct, \\ \eta = x - ct. \end{cases}$$
(2.4.1)

**Proposition 2.4.1.** The following equality holds:

$$u_{tt} - c^2 u_{xx} = -4c^2 u_{\xi\eta}.$$
 (2.4.2)

*Proof.* From (2.4.1) we see that  $x = \frac{1}{2}(\xi + \eta)$  and  $t = \frac{1}{2c}(\xi - \eta)$  and therefore due to chain rule  $v_{\xi} = \frac{1}{2}v_x + \frac{1}{2c}v_t$  and  $v_{\eta} = \frac{1}{2}v_x - \frac{1}{2c}v_t$  and therefore

$$-4c^2u_{\xi\eta} = -(c\partial_x + \partial_t)(c\partial_x - \partial_t)u = u_{tt} - c^2u_{xx}$$

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Therefore wave equation (2.3.1) becomes in the characteristic coordinates

$$u_{\xi\eta} = 0 \tag{2.4.3}$$

which we rewrite as  $(u_{\xi})_{\eta} = 0 \implies u_{\xi} = \phi'(\xi)$  (really,  $u_{\xi}$  should not depend on  $\eta$  and it is convenient to denote by  $\phi(\xi)$  the primitive of  $u_{\xi}$ ). Then  $(u - \phi(\xi))_{\xi} = 0 \implies u - \phi(\xi) = \psi(\eta)$  (due to the same arguments) and therefore

$$u = \phi(\xi) + \psi(\eta) \tag{2.4.4}$$

is the general solution to (2.4.3).

### 2.4.2 Application of characteristic coordinates

Example 2.4.1. Consider Goursat problem for (2.4.3):

$u_{\xi\eta} = 0$	as $\xi > 0, \eta > 0$
$u _{\eta=0} = g(\xi)$	as $\xi > 0$ ,
$u _{\xi=0} = h(\eta)$	as $\eta > 0$

where g and h must satisfy compatibility condition g(0) = h(0) (really g(0) = u(0,0) = h(0)).

Then one can see easily that  $u(\xi, \eta) = g(\xi) + h(\eta) - g(0)$  solves Goursat problem. Plugging (2.4.1) into (2.4.4) we get for a general solution to (2.3.1)

$$u = \phi(x + ct) + \psi(x - ct)$$
 (2.4.5)

which is exactly (2.3.7).

### 2.4.3 D'Alembert formula

So far we achieved nothing new. Consider now IVP:

$$u_{tt} - c^2 u_{xx} = f(x, t), (2.4.6)$$

$$u|_{t=0} = g(x), \tag{2.4.7}$$

$$u_t|_{t=0} = h(x). \tag{2.4.8}$$

It is convenient for us to assume that g = h = 0. Later we will get rid of this assumption. Rewriting (2.4.6) as

$$\tilde{u}_{\xi\eta} = -\frac{1}{4c^2}\tilde{f}(\xi,\eta)$$

(where  $\tilde{u}$  etc means that we use characteristic coordinates) we get after integration

$$\tilde{u}_{\xi} = -\frac{1}{4c^2} \int^{\eta} \tilde{f}(\xi, \eta') \, d\eta' = -\frac{1}{4c^2} \int_{\xi}^{\eta} \tilde{f}(\xi, \eta') \, d\eta' + \phi'(\xi)$$

with an indefinite integral in the middle.

Note that t = 0 means exactly that  $\xi = \eta$ , but then  $u_{\xi} = 0$  there. Really,  $u_{\xi}$  is a linear combination of  $u_t$  and  $u_x$  but both of them are 0 as t = 0. Therefore  $\phi'(\xi) = 0$  and

$$\tilde{u}_{\xi} = \frac{1}{4c^2} \int_{\eta}^{\xi} \tilde{f}(\xi, \eta') \, d\eta'$$

where we flipped limits and changed sign.

Integrating with respect to  $\xi$  we arrive to

$$\tilde{u} = \frac{1}{4c^2} \int^{\xi} \left[ \int_{\eta}^{\xi'} \tilde{f}(\xi', \eta') \, d\eta' \right] d\xi' = \frac{1}{4c^2} \int_{\eta}^{\xi} \left[ \int_{\eta}^{\xi'} \tilde{f}(\xi', \eta') \, d\eta' \right] d\xi' + \psi(\eta)$$

and  $\psi(\eta)$  also must vanish because u = 0 as t = 0 (i.e.  $\xi = \eta$ ). So

$$\tilde{u}(\xi,\eta) = \frac{1}{4c^2} \int_{\eta}^{\xi} \left[ \int_{\eta}^{\xi'} \tilde{f}(\xi',\eta') \, d\eta' \right] d\xi'.$$
(2.4.9)

We got a solution as a *double integral* but we want to write it down as 2-dimensional integral

$$\tilde{u}(\xi,\eta) = \frac{1}{4c^2} \iint_{\tilde{\Delta}(\xi,\eta)} \tilde{f}(\xi',\eta') \, d\eta' d\xi'.$$
(2.4.10)

But what is  $\Delta$ ? Consider  $\xi > \eta$ . Then  $\xi'$  should run from  $\eta$  to  $\xi$  and for fixed  $\xi'$ ,  $\eta < \xi' < \xi$  eta should run from  $\eta$  to  $\xi'$ . So, we get a triangle bounded by  $\xi' = \eta'$ ,  $\xi' = \xi$  and  $\eta' = \eta$ :



But in coordinates (x, t) this domain  $\Delta(x, t)$  is bounded by t = 0 and two characteristics passing through (x, t):

So, we get

$$u(x,t) = \frac{1}{2c} \iint_{\Delta(x,t)} f(x',t') \, dx' dt'.$$
(2.4.11)

because we need to replace  $d\xi' d\eta'$  by |J| dx' dt' with Jacobian J.

*Exercise* 2.4.1. Calculate J and justify factor 2c.

Remark 2.4.1. (a) Formula (2.4.11) solves IVP for inhomogeneous wave equation with homogeneous (that means equal 0) initial data, while (2.3.14) solved IVP for homogeneous equation with inhomogeneous initial data. Due to linearity we will be able to combine those formulae!

(b) Both (2.4.11) and (2.3.14) are valid for t > 0 and for t < 0.

*Example 2.4.2.* (a) Find solution u(x,t) to

$$4u_{tt} - 9u_{xx} = \frac{x}{t+1} \qquad -\infty < x < \infty, \ -1 < t < \infty, \qquad (2.4.12)$$

$$u|_{t=0} = 0, \quad u_t|_{t=0} = 0.$$
 (2.4.13)

(b) If (2.4.12)–(2.4.13) are fulfilled for -3 < x < 3 only, where solution is uniquely defined?

Solution. (a) Using D'Alembert formula

$$\begin{split} u(x,t) &= \frac{1}{12} \int_0^t \int_{x-\frac{3}{2}(t-\tau)/2}^{x+3(t-\tau)/2} \frac{\xi}{\tau+1} d\xi d\tau \\ &= \frac{1}{24} \int_0^t \frac{\left(x+\frac{3}{2}(t-\tau)\right)^2 - \left(x-\frac{3}{2}(t-\tau)\right)^2}{\tau+1} d\tau \\ &= \frac{1}{4} \int_0^t \frac{x(t-\tau)}{\tau+1} d\tau \\ &= \frac{1}{4} x \big( (t+1) \ln(t+1) - t \big). \end{split}$$

(b) Solution is uniquely defined at points (x, t) such that the base of the characteristic triangle is contained in [-3, 3]:



*Example 2.4.3.* Find solution u(x, t) to

$$u_{tt} - 4u_{xx} = \sin(x)\sin(2t) \qquad -\infty < x < \infty, \ -1 < t < \infty, \ (2.4.14)$$
$$u_{t=0} = 0, \ u_t|_{t=0} = 0. \tag{2.4.15}$$

Solution. Using D'Alembert formula

$$u(x,t) = \frac{1}{4} \int_0^t \int_{x-2(t-\tau)}^{x+2(t-\tau)} \sin(\xi) \sin(2\tau) \, d\xi d\tau$$
  
=  $\frac{1}{4} \int_0^t \sin(2\tau) \Big( \cos(x-2t+2\tau) - (\cos(x+2t\tau 2-\tau)) \Big) \, d\tau$   
=  $\frac{1}{2} \int_0^t \sin(2\tau) \sin(x) \sin(2t-2\tau) \, d\tau$ 

where we used  $\cos(\alpha) - \cos(\beta) = 2\sin\left(\frac{\beta+\alpha}{2}\right)\sin\left(\frac{\beta-\alpha}{2}\right)$ . Then

$$u(x,t) = \frac{1}{2} \int_0^t \sin(2\tau) \sin(x) \sin(2t - 2\tau) d\tau$$
  
=  $\sin(x) \int_0^t \left( \cos(2t - 4\tau) - \cos(2t) \right) d\tau$   
=  $\sin(x) \left[ -\frac{1}{4} \sin(2t - 4\tau) - \tau \cos(2t) \right]_{\tau=0}^{\tau=t}$   
=  $\sin(x) \left[ \frac{1}{2} \sin(2t) - t \cos(2t) \right].$ 

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### Problems to Section 2.4

Problem 1. Solve IVP

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x, t); \\ u|_{t=0} = g(x), \\ u_t|_{t=0} = h(x) \end{cases}$$

with

$$f(x,t) = \sin(\alpha x), \qquad g(x) = 0, \qquad h(x) = 0;$$
  

$$f(x,t) = \sin(\alpha x)\sin(\beta t), \qquad g(x) = 0; \qquad h(x) = 0;$$
  

$$f(x,t) = f(x), \qquad g(x) = 0, \qquad h(x) = 0; \qquad (a)$$
  

$$f(x,t) = f(x)t, \qquad g(x) = 0, \qquad h(x) = 0, \qquad (b)$$

in the case (a) assume that f(x) = F''(x) and in the case (b) assume that f(x) = F'''(x).

Problem 2. Find formula for solution of the Goursat problem

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x, t), & x > c|t|, \\ u|_{x=-ct} = g(t), & t < 0, \\ u|_{x=ct} = h(t), & t > 0 \end{cases}$$

provided g(0) = h(0).

*Hint.* Contribution of the right-hand expression will be

$$-\frac{1}{4c^2}\iint_{R(x,t)}f(x',t')\,dx'dt'$$

with  $R(x,t) = \{(x',t'): 0 < x' - ct' < x - ct, 0 < x' + ct' < x + ct\}.$ 

 $Problem \ 3.$  Find the general solutions of the following equations:

$$\begin{cases} u_{xy} = u_x u_y u^{-1}; \\ u_{xy} = u_x u_y; \\ u_{xy} = \frac{u_x u_y u}{u^2 + 1}; \end{cases}$$

Problem 4. (a) Find solution u(x,t) to

$$\begin{cases} u_{tt} - u_{xx} = (x^2 - 1)e^{-\frac{x^2}{2}}, \\ u_{t=0} = -e^{-\frac{x^2}{2}}, \quad u_t|_{t=0} = 0. \end{cases}$$

(b) Find  $\lim_{t\to+\infty} u(x,t)$ .