Chapter 5

Fourier transform

In this Chapter we consider Fourier transform which is the most useful of all integral transforms.

5.1 Fourier transform, Fourier integral

5.1.1 Heuristics

In Section 4.5 we wrote Fourier series in the complex form

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{i\pi n x}{l}} \tag{5.1.1}$$

with

$$c_n = \frac{1}{2l}\int_{-l}^{l} f(x) e^{-\frac{i\pi n x}{l}}\, dx \qquad n = \ldots, -2, -1, 0, 1, 2, \ldots \tag{5.1.2}$$

and

$$2l \sum_{n=-\infty}^{\infty} |c_n|^2 = \int_{-l}^{l} |f(x)|^2\, dx. \tag{5.1.3}$$

From this form we *formally*, without any justification, will deduct Fourier integral.

First we introduce

$$k_n := \frac{\pi n}{l} \qquad \text{and} \qquad \Delta k_n = k_n - k_{n-1} = \frac{\pi}{l} \tag{5.1.4}$$

and rewrite (5.1.1) as

$$f(x) = \sum_{n=-\infty}^{\infty} C(k_n)e^{ik_nx}\Delta k_n \tag{5.1.5}$$

with

$$C(k) = \frac{1}{2\pi}\int_{-l}^{l} f(x)e^{-ikx}\,dx \tag{5.1.6}$$

where we used $C(k_n) \coloneqq c_n/(\Delta k_n)$; equality (5.1.3)

$$2\pi \sum_{n=-\infty}^{\infty} |C(k_n)|^2\Delta k_n. \tag{5.1.7}$$

Now we *formally* set $l\to +\infty$; then integrals from $-l$ to l in the right-hand expression of (5.1.6) and the left-hand expression of (5.1.7) become integrals from $-\infty$ to $+\infty$.

Meanwhile, $\Delta k_n \to +0$ and *Riemannian sums* in the right-hand expressions of (5.1.5) and (5.1.7) become integrals:

$$f(x) = \int_{-\infty}^{\infty} C(k)e^{ikx}\,dk \tag{5.1.8}$$

with

$$C(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx; \tag{5.1.9}$$

Meanwhile (5.1.3) becomes

$$\int_{-\infty}^{\infty} |f(x)|^2\,dx = 2\pi\int_{-\infty}^{\infty} |C(k)|^2\,dk. \tag{5.1.10}$$

5.1.2 5.1.2. Definitions and remarks

Definition 5.1.1. (a) Formula (5.1.9) gives us a *Fourier transform* of $f(x)$, it usually is denoted by "hat":

$$\hat{f}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx; \tag{FT}$$

sometimes it is denoted by "tilde" ($\tilde{f}$), and seldom just by a corresponding capital letter $F(k)$.

(b) Expression (5.1.8) is a *Fourier integral* a.k.a. *inverse Fourier transform*:

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(k)e^{ikx}\,dk \tag{FI}$$

a.k.a.

$$\check{F}(x) = \int_{-\infty}^{\infty} F(k)e^{ikx}\,dk \tag{IFT}$$

Remark 5.1.1. Formula (5.1.10) is known as *Plancherel theorem*

$$\int_{-\infty}^{\infty} |f(x)|^2\,dx = 2\pi \int_{-\infty}^{\infty} |\hat{f}(k)|^2\,dk. \tag{PT}$$

Remark 5.1.2. (a) Sometimes exponents of $\pm ikx$ is replaced by $\pm 2\pi ikx$ and factor $1/(2\pi)$ dropped.

(b) Sometimes factor $\frac{1}{\sqrt{2\pi}}$ is placed in both Fourier transform and Fourier integral:

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx; \tag{FT*}$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k)e^{ikx}\,dk. \tag{FI*}$$

Then FT and IFT differ only by i replaced by $-i$ and Plancherel theorem becomes

$$\int_{-\infty}^{\infty} |f(x)|^2\,dx = \int_{-\infty}^{\infty} |\hat{f}(k)|^2\,dk. \tag{PT*}$$

In this case Fourier transform and inverse Fourier transform differ only by $-i$ instead of i (very symmetric form) and both are *unitary operators*. In all assignments indicate which form of F.T. you use!

Remark 5.1.3. We can consider corresponding operator $LX = -X''$ in the space $L^2(\mathbb{R})$ of the square integrable functions on $\mathbb{R}$.

(a) However, e^{ikx} are no more eigenfunctions since they do not belong to this space. Correspondingly, k^2 *with* $k \in \mathbb{R}$ *are not eigenvalues*. In advanced Real Analysis such functions and numbers often are referred as *generalized eigenfunctions* and *generalized eigenvalues*.

(b) The generalized eigenvalues which are not eigenvalues form *continuous spectrum* in contrast to *point spectrum* formed by eigenvalues.

(c) Physicists often omit word "generalized".

(d) More details in Section 13.4.

Remark 5.1.4. (a) For justification see Appendix 5.1.3.

(b) Pointwise convergence of Fourier integral is discussed in Section 5.1.3.

(c) Multidimensional Fourier transform and Fourier integral are discussed in Appendix 5.2.5.

(d) Fourier transform in the complex domain (for those who took "Complex Variables") is discussed in Appendix 5.2.5.

(e) Fourier Series interpreted as Discrete Fourier transform are discussed in Appendix 5.2.5.

5.1.3 cos- and sin-Fourier transform and integral

Applying the same arguments as in Section 4.5 we can rewrite formulae (5.1.8)–(5.1.10) as

$$f(x) = \int_0^\infty \bigl(A(k)\cos(kx) + B(k)\sin(kx)\bigr)\,dk \tag{5.1.11}$$

with

$$A(k) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos(kx)\,dx, \tag{5.1.12}$$

$$B(k) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin(kx)\,dx, \tag{5.1.13}$$

and

$$\int_{-\infty}^{\infty} |f(x)|^2\,dx = \pi\int_0^\infty \bigl(|A(k)|^2 + |B(k)|^2\bigr)\,dk. \tag{5.1.14}$$

$A(k)$ and $B(k)$ are cos- and sin- Fourier transforms and

(a) $f(x)$ is even function iff $B(k) = 0$.

(b) $f(x)$ is odd function iff $A(k)=0$.

Therefore

(a) Each function on $[0,\infty)$ could be decomposed into cos-Fourier integral

$$f(x)=\int_0^\infty A(k)\cos(kx)\,dk \tag{5.1.15}$$

with

$$A(k)=\frac{2}{\pi}\int_0^\infty f(x)\cos(kx)\,dx. \tag{5.1.16}$$

(b) Each function on $[0,\infty)$ could be decomposed into sin-Fourier integral

$$f(x)=\int_0^\infty B(k)\sin(kx)\,dk \tag{5.1.17}$$

with

$$B(k)=\frac{2}{\pi}\int_0^\infty f(x)\sin(kx)\,dx. \tag{5.1.18}$$

Appendix 5.1.A. Justification

Let $u(x)$ be smooth fast decaying function; let us decompose it as in Section 4.B (but now we are in the simpler 1-dimensional framework and $\Gamma=2\pi\mathbb{Z}$):

$$u(x)=\int_0^1 u(k;x)\,dk \tag{5.1.A.1}$$

with

$$u(k;x)=\sum_{m=-\infty}^{\infty} e^{-2\pi kmi}u(x+2\pi m). \tag{5.1.A.2}$$

Here $u(k;x)$ is *quasiperiodic with quasimomentum* k

$$u(k;x+2\pi n)=e^{2\pi nki}u(k;x)\qquad \forall n\in\mathbb{Z}\ \forall x\in\mathbb{R}. \tag{5.1.A.3}$$

Indeed,

$$u(k;x+2\pi n)=\sum_{m=-\infty}^{\infty} e^{-2\pi kmi}u(x+2\pi(m+n)) \overset{m:=m+n}{=} \\ \sum_{m=-\infty}^{\infty} e^{-2\pi k(m-n)i}u(x+2\pi m)=e^{2\pi nki}u(k;x).$$

Then $e^{-ikx}u(k;x)$ is periodic and one can decompose it into Fourier series

$$u(k;x)=\sum_{n=-\infty}^{\infty} e^{inx}e^{ikx}c_n(k)=\sum_{n=-\infty}^{\infty} e^{i(n+k)x}c_n(k) \tag{5.1.A.4}$$

(where we restored $u(k;x)$ multiplying by e^{ikx}) with

$$c_n(k)=\frac{1}{2\pi}\int_0^{2\pi} e^{-i(n+k)x}u(k;x)\,dx \tag{5.1.A.5}$$

and

$$2\pi\sum_{n=-\infty}^{\infty}|c_n(k)|^2=\int_0^{2\pi}|u(k;x)|^2\,dx. \tag{5.1.A.6}$$

Plugging (5.1.A.4) into (5.1.A.1), that is

$$u(k;x)=\sum_{n=-\infty}^{\infty} e^{i(n+k)x}c_n(k), \qquad u(x)=\int_0^1 u(k;x)\,dk,$$

we get

$$u(x)=\int_0^1 \sum_{n=-\infty}^{\infty} c_n(k)e^{i(n+k)x}\,dk=\sum_{n=-\infty}^{\infty}\int_n^{n+1} C(\omega)e^{i\omega x}\,d\omega$$
$$=\int_{-\infty}^{\infty} C(\omega)e^{i\omega x}\,d\omega$$

where $C(\omega):=c_n(k)$ with $n=\lfloor\omega\rfloor$ and $k=\omega-\lfloor\omega\rfloor$ which are respectively integer and fractional parts of ω. So, we got decomposition of $u(x)$ into Fourier integral.

Next, plugging (5.1.A.2) into (5.1.A.5), that is

$$u(k;x)=\sum_{m=-\infty}^{\infty} e^{-2\pi kmi}u(x+2\pi m), \quad c_n(k)=\frac{1}{2\pi}\int_0^{2\pi} e^{-i(n+k)x}u(k;x)\,dx,$$

we get

$$C(\omega)=\frac{1}{2\pi}\int_0^{2\pi} e^{-i\omega x}\sum_{m=-\infty}^{\infty} e^{-2\pi ikm}u(x+2\pi m)\,dx=$$
$$\frac{1}{2\pi}\int_{2\pi m}^{2\pi(m+1)} e^{-i\omega y}u(y)\,dy=\int_{-\infty}^{2\pi} e^{-i\omega y}u(y)\,dy$$

where we set $y = x + 2\pi m$. So, we got exactly formula for Fourier transform.

Finally, (5.1.A.6), that is

$$2\pi \sum_{n=-\infty}^{\infty} |c_n(k)|^2 = \int_0^{2\pi} |u(k;x)|^2\,dx,$$

implies

$$2\pi \sum_{n=-\infty}^{\infty} \int_0^1 |c_n(k)|^2\,dk = \int_0^{2\pi} \Bigl(\int_0^1 |u(k;x)|^2\,dk\Bigr)\,dx$$

where the left hand expression is exactly

$$2\pi \sum_{n=-\infty}^{\infty} \int_n^{n+1} |C(\omega)|^2\,d\omega = 2\pi \int_{-\infty}^{\infty} |C(\omega)|^2\,d\omega$$

and the right hand expression is

$$\int_0^{2\pi} \Bigl(\int_0^1 \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} e^{2\pi i k(l-m)} u(x+2\pi m)\bar{u}(x+2\pi l)\,dk\Bigr)\,dx$$

and since $\int_0^1 e^{2\pi i k(l-m)}\,dk = \delta_{lm}$ (1 as $l = m$ and 0 otherwise) it is equal to

$$\int_0^{2\pi} \sum_{m=-\infty}^{\infty} |u(x+2\pi m)|^2\,dx = \int_{-\infty}^{\infty} |u(x)|^2\,dx.$$

So, we arrive to Plancherel theorem.

Appendix 5.1.B. Discussion: pointwise convergence of Fourier integrals and series

Recall Theorem 4.4.4: Let f be a piecewise continuously differentiable function. Then the Fourier series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \Bigl(a_n \cos(\frac{\pi n x}{l}) + a_n \cos(\frac{\pi n x}{l})\Bigr) \tag{5.1.B.1}$$

converges to $\frac{1}{2}\bigl(f(x+0)+f(x-0)\bigr)$ if x is internal point and f is discontinuous at x.

Exactly the same statement holds for Fourier Integral in the real form

$$\int_0^\infty \Bigl(A(k)\cos(kx) + B(k)\sin(kx)\Bigr)\,dk \tag{5.1.B.2}$$

where $A(k)$ and $B(k)$ are cos-and sin-Fourier transforms.

None of them however holds for Fourier series or Fourier Integral in the complex form:

$$\sum_{n=-\infty}^{\infty} c_n e^{i\frac{\pi n x}{l}}, \tag{5.1.B.3}$$

$$\int_{-\infty}^{\infty} C(k)e^{ikx}\,dk. \tag{5.1.B.4}$$

Why and what remedy do we have? If we consider definition of the partial sum of (5.1.B.1) and then rewrite in the complex form and similar deal with (5.1.B.4) we see that in fact we should look at

$$\lim_{N\to\infty} \sum_{n=-N}^{N} c_n e^{i\frac{\pi n x}{l}}, \tag{5.1.B.5}$$

$$\lim_{N\to\infty} \int_{-N}^{N} C(k)e^{ikx}\,dk. \tag{5.1.B.6}$$

Meanwhile convergence in (5.1.B.3) and (5.1.B.4) means more than this:

$$\lim_{M,N\to\infty} \sum_{n=-M}^{N} c_n e^{i\frac{\pi n x}{l}}, \tag{5.1.B.7}$$

$$\lim_{M,N\to\infty} \int_{-M}^{N} C(k)e^{ikx}\,dk \tag{5.1.B.8}$$

where M, N tend to ∞ independently. So the remedy is simple: understand convergence as in (5.1.B.5), (5.1.B.6) rather than as in (5.1.B.7), (5.1.B.8).

For integrals such limit is called principal value of integral and is denoted by

$$\operatorname{pv}\int_{-\infty}^{\infty} G(k)\,dk.$$

Also similarly is defined the principal value of the integral, divergent in the finite point

$$\operatorname{pv}\int_a^b G(k)\,dk := \lim_{\varepsilon\to+0}\Bigl(\int_a^{c-\varepsilon} G(k)\,dk + \int_{c+\varepsilon}^b G(k)\,dk\Bigr)$$

if there is a singularity at $c\in(a,b)$. Often instead of pv is used original (due to Cauchy) vp (valeur principale) and some other notations.

This is more general than the *improper integrals* studied in the end of Calculus I (which in turn generalize Riemann integrals). Those who took Complex Variables encountered such notion.

5.2 Properties of Fourier transform

5.2.1 Basic properties

In the previous Section 5.1 we introduced Fourier transform and Inverse Fourier transform

$$\hat{f}(k) = \frac{\kappa}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx \tag{FT}$$

$$\check{F}(x) = \frac{1}{\kappa}\int_{-\infty}^{\infty} F(k)e^{ikx}\,dk \tag{IFT}$$

with $\kappa = 1$ (but here we will be a bit more flexible):

Theorem 5.2.1. $F=\hat{f} \iff f=\check{F}$.

Proof. Already "proved" (formally). □

Theorem 5.2.2. *(i) Fourier transform: $f\mapsto\hat{f}$ is a linear operator $L^2(\mathbb{R},\mathbb{C})\to L^2(\mathbb{R},\mathbb{C})$;*

(ii) Inverse Fourier transform: $F\mapsto\check{F}$ is an inverse operator (and also a linear operator) $L^2(\mathbb{R},\mathbb{C})\to L^2(\mathbb{R},\mathbb{C})$;

(iii) If $\kappa=\sqrt{2\pi}$ these operators are unitary *i.e. preserve norm and an inner product:*

$$\|f\| = \Bigl(\int_{\mathbb{R}} |f(x)|^2\,dx\Bigr)^{\frac{1}{2}}, \tag{5.2.1}$$

$$(f,g) = \int_{\mathbb{R}} f(x)\bar{g}(x)\,dx. \tag{5.2.2}$$

Proof. Easy. Preservation of inner product follows from preservation of norm. □

Remark 5.2.1. (a) Here $L^2(\mathbb{R},\mathbb{C})$ is a space of square integrable complexample-valued functions. Accurate definition requires a measure theory (studied in the course of Real Analysis). Alternatively one can introduce this space as a closure of the set of square integrable continuous functions but it also require a certain knowledge of Real Analysis.

(b) Properties (i) and (ii)are obvious and (iii) is due to Plancherel's theorem.

(c) In Quantum Mechanics Fourier transform is sometimes referred as "going to p-representation" (a.k.a. momentum representation) and Inverse Fourier transform is sometimes referred as "going to q-representation" (a.k.a. coordinate representation). In this case $\pm ikx$ is replaced by $\pm i\hbar^{-1}kx$ and 2π by $2\pi\hbar$.

Theorem 5.2.3. *(i)* $g(x) = f(x-a) \implies \hat{g}(k) = e^{-ika}\hat{f}(k)$;

(ii) $g(x) = f(x)e^{ibx} \implies \hat{g}(k) = \hat{f}(k-b)$;

(iii) $g(x) = f'(x) \implies \hat{g}(k) = ik\hat{f}(k)$;

(iv) $g(x) = xf(x) \implies \hat{g}(k) = i\hat{f}'(k)$;

(v) $g(x) = f(\lambda x) \implies \hat{g}(k) = |\lambda|^{-1}\hat{f}(\lambda^{-1}k)$;

Proof. Here for brevity we do not write that all integrals are over $\mathbb{R}$ and set $\kappa = 2\pi$.

(i) $\hat{g} = \int e^{-ikx}g(x)\,dx = \int e^{-ikx}f(x-a)\,dx = \int e^{-ik(x+a)}f(x)\,dx = e^{-ika}\hat{f}(k)$. We replaced x by $(x+a)$ in the integral.

(ii) $\hat{g} = \int e^{-ikx}g(x)\,dx = \int e^{-ikx}e^{ibx}f(x)\,dx = \int e^{-i(k-b)x}f(x)\,dx = \hat{f}(k-b)$.

(iii) $\hat{g} = \int e^{-ikx}g(x)\,dx = \int e^{-ikx}f'(x)\,dx \overset{\text{by parts}}{=} -\int \bigl(e^{-ikx}\bigr)'f(x)\,dx = ik\hat{f}(k)$.

(iv) $\hat{g} = \int e^{-ikx}g(x)\,dx = \int e^{-ikx}xf(x)\,dx = \int i\partial_k\bigl(e^{-ikx}\bigr)\,f(x)\,dx = i\hat{f}'(k)$.

(v) $\hat{g} = \int e^{-ikx}g(x)\,dx = \int e^{-ikx}f(\lambda x)\,dx = \int e^{-ik\lambda^{-1}x}f(x)\,|\lambda|^{-1}dx = |\lambda|^{-1}\hat{f}(\lambda^{-1}k)$. Here we replaced x by $\lambda^{-1}x$ in the integral and $|\lambda|^{-1}$ is an absolute value of Jacobian.

□

Remark 5.2.2. 1. Differentiating (i) by a and taking $a = 0$ we get $-\widehat{(f')}(k) = -ik\hat{f}(k)$ (another proof of (iii).

2. Differentiating (ii) by b and taking $b = 0$ we get $i\widehat{(xf)}(k) = -\hat{f}'(k)$ (another proof of (iv).

Corollary 5.2.4. *f is even (odd) iff $\hat{f}$ is even (odd).*

5.2.2 Convolution

Definition 5.2.1. *Convolution* of functions f and g is a function $f * g$:

$$(f * g)(x) := \int f(x-y)g(y)\,dy. \tag{5.2.3}$$

Theorem 5.2.5. *(i) $h = f * g \implies \hat{h}(k) = \frac{2\pi}{\kappa}\hat{f}(k)\hat{g}(k)$;*

(ii) $h(x) = f(x)g(x) \implies \hat{h} = \kappa\hat{f} * \hat{g}$;

Proof. Again for brevity we do not write that all integrals are over $\mathbb{R}$ and set $\kappa = 2\pi$.

(i) Obviously, for $h = f * g$, that is $h(x) = \int f(x-y)g(y)\,dy$,

$$\hat{h}(k) = \frac{\kappa}{2\pi}\int e^{-ixk}h(x)\,dx = \frac{\kappa}{2\pi}\iint e^{-ixk}f(x-y)g(y)\,dxdy.$$

replacing in the integral $x := y + z$ we arrive to

$$\frac{\kappa}{2\pi}\iint e^{-i(y+z)k}f(z)g(y)\,dzdy = \frac{\kappa}{2\pi}\int e^{-izk}f(z)\,dz \times \int e^{-iyk}g(y)\,dz$$

which is equal to $\frac{2\pi}{\kappa}\hat{f}(k)\hat{g}(k)$.

(ii) Similarly, using inverse Fourier transform we prove that $\hat{f} * \hat{g}$ is a Fourier transform of $\frac{\kappa_1}{2\pi}fg$ where $\kappa_1 = \frac{2\pi}{\kappa}$

□

5.2.3 Examples

Example 5.2.1. (a) Let $f(x)=\begin{cases} e^{-\alpha x} & x>0,\\ 0 & x<0.\end{cases}$ Here $\operatorname{Re}(\alpha)>0$.

$$\hat{f}(k)=\int_0^\infty e^{-(\alpha+ik)x}\,dx=-\frac{\kappa}{2\pi(\alpha+ik)}e^{-(\alpha+ik)x}\Bigr|_{x=0}^{x=\infty}=\frac{\kappa}{2\pi(\alpha+ik)}.$$

(b) Correspondingly, $f(x)=\begin{cases} e^{\alpha x} & x<0,\\ 0 & x>0\end{cases} \implies \hat{f}(k)=\dfrac{\kappa}{2\pi(\alpha+ik)}.$

(c) Then $f(x)=e^{-\alpha|x|} \implies \hat{f}(k)=\dfrac{\kappa\alpha}{\pi(\alpha^2+k^2)}.$

Example 5.2.2. Let $f(x)=\frac{1}{\alpha^2+x^2}$ with $\operatorname{Re}(\alpha)>0$. Then $\hat{f}(k)=\kappa e^{-\alpha|k|}$. Indeed, using Example 5.2.1(c) we conclude that $f(x)$ is an inverse Fourier transform of $\kappa e^{-\alpha|k|}$ since we need only to take into account different factors and replace i by $-i$ (for even/odd functions the latter step could be replaced by multiplication by ± 1).

Using Complex Variables one can calculate it directly (residue should be calculated at $-\alpha i\operatorname{sign}(k)$).

Example 5.2.3. Let $f(x)=e^{-\frac{\alpha}{2}x^2}$ with $\operatorname{Re}(\alpha)\ge 0$. Here even for $\operatorname{Re}(\alpha)=0$ Fourier transform exists since integrals are converging albeit not absolutely.

Note that $f'=-\alpha x f$. Applying Fourier transform and Theorem 5.2.3 (iii), (iv) to the left and right expressions, we get $ik\hat{f}=-i\alpha\hat{f}'$; solving it we arrive to $\hat{f}=Ce^{-\frac{1}{2\alpha}k^2}$.

To find C note that $C=\hat{f}(0)=\frac{\kappa}{2\pi}\int e^{-\frac{\alpha}{2}x^2}\,dx$ and for real $\alpha>0$ we make a change of variables $x=\alpha^{-\frac{1}{2}}z$ and arrive to $C=\frac{\kappa}{\sqrt{2\pi\alpha}}$ because $\int e^{-z^2/2}\,dz=\sqrt{2\pi}$. Therefore

$$\hat{f}(k)=\frac{\kappa}{\sqrt{2\pi\alpha}}e^{-\frac{1}{2\alpha}k^2}.$$

Remark 5.2.3. Knowing Complex Variables one can justify it for complex α with $\operatorname{Re}(\alpha)\ge 0$; we take a correct branch of $\sqrt{\alpha}$ (condition $\operatorname{Re}(\alpha)\ge 0$ prevents going around origin).

In particular, $(\pm i)^{\frac{1}{2}}=e^{\pm\frac{i\pi}{4}}=\frac{1}{\sqrt{2}}(1\pm i)$ and therefore for $\alpha=\pm i\beta$ with for $\beta>0$ we get $f=e^{\mp\frac{i\beta}{2}x^2}$ and

$$\hat{f}(k)=\frac{\kappa}{\sqrt{2\pi\beta}}e^{\pm\frac{\pi i}{4}}e^{\pm\frac{i}{2\beta}k^2}=\frac{\kappa}{2\sqrt{\pi\beta}}(1\mp i)e^{\pm\frac{i}{2\beta}k^2}.$$

5.2.4 Poisson summation formula

Theorem 5.2.6. *Let $f(x)$ be a continuous function on the line $(-\infty,\infty)$ which vanishes for large $|x|$. Then for any $a>0$*

$$\sum_{n=-\infty}^{\infty} f(an) = \sum_{n=-\infty}^{\infty} \frac{2\pi}{a}\hat{f}(\frac{2\pi}{a}n). \tag{5.2.4}$$

Proof. Observe that function

$$g(x) = \sum_{n=-\infty}^{\infty} f(x+an)$$

is periodic with period a. Note that the Fourier coefficients of $g(x)$ on the interval $(-\frac{a}{2},\frac{a}{2})$ are $b_m = \frac{2\pi}{a}\hat{f}(\frac{2\pi}{a})$, where $\hat{f}(k)$ is the Fourier transform of $f(x)$.

Finally, in the Fourier series of $g(x)$ on $(-\frac{a}{2},\frac{a}{2})$ plug $x=0$ to obtain $g(0)=\sum_m b_m$ which coincides with (5.2.4). □

5.2.5 Final remarks

Remark 5.2.4. (a) Properties of Multidimensional Fourier transform and Fourier integral are discussed in Appendix 5.2.5.

(b) It is *very important* to do all problems from Problems to Sections 5.1 and 5.2: instead of calculating Fourier transforms directly you use Theorem 5.2.3 to expand the "library" of Fourier transforms obtained in Examples 5.2.1–5.2.3.

Appendix 5.2.A. Multidimensional Fourier transform, Fourier integral

Definition 5.2.A.1. Multidimensional Fourier transform is defined as

$$\hat{f}(\boldsymbol{k}) = \Bigl(\frac{\kappa}{2\pi}\Bigr)^n \iiint_{\mathbb{R}^n} f(\boldsymbol{x})e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\,d^n x \tag{MFT}$$

$$\check{F}(\boldsymbol{x}) = \Bigl(\frac{1}{\kappa}\Bigr)^n \iiint_{\boldsymbol{R}^n} F(\boldsymbol{k})e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\,d^n \boldsymbol{k} \tag{MIFT}$$

with $\kappa=1$ (but here we will be a bit more flexible).

All the main properties of 1-dimensional Fourier transform are preserved (with obvious modifications) but some less obvious modifications are mentioned:

Remark 5.2.A.1. Theorem 5.2.3(v) is replaced by

$$g(\boldsymbol{x}) = f(Q\boldsymbol{x}) \implies \hat{g}(\boldsymbol{k}) = |\det Q|^{-1}\hat{f}(Q^{*\,-1}\boldsymbol{k}) \tag{5.2.A.1}$$

where Q is a non-degenerate linear transformation.

Remark 5.2.A.2. Example 5.2.2 is replaced by the following:

(a) Let $f(x) = e^{-\frac{1}{2}A\boldsymbol{x}\cdot\boldsymbol{x}}$ where A is a symmetric (but not necessarily real matrix) $A^T = A$ with positive definite real part:

$$\operatorname{Re}(A\boldsymbol{x}\cdot\boldsymbol{x}) \ge \epsilon|\boldsymbol{x}|^2 \qquad \forall \boldsymbol{x} \tag{5.2.A.2}$$

with $\epsilon > 0$. One can prove that inverse matrix A^{-1} has the same property and

$$\hat{f}(\boldsymbol{k}) = \Bigl(\frac{\kappa}{\sqrt{2\pi}}\Bigr)^n \det(A^{-\frac{1}{2}})e^{-\frac{1}{2}A^{-1}\boldsymbol{k}\cdot\boldsymbol{k}}. \tag{5.2.A.3}$$

(b) As long as a complex matrix A does not have eigenvalues on $(-\infty, 0]$, one can define properly A^z, $\log(A)$ etc.

(c) In the case of Hermitian matrix A we have

$$\hat{f}(\boldsymbol{k}) = \Bigl(\frac{\kappa}{\sqrt{2\pi}}\Bigr)^n |\det A|^{-\frac{1}{2}}e^{-\frac{1}{2}A^{-1}\boldsymbol{k}\cdot\boldsymbol{k}}. \tag{5.2.A.4}$$

(d) This could be generalized to the matrices satisfying condition

$$\operatorname{Re}(A\boldsymbol{x}\cdot\boldsymbol{x}) \ge 0 \qquad \forall \boldsymbol{x}, \qquad \det A \ne 0 \tag{5.2.A.5}$$

rather than (5.2.A.2). In particular, for anti-Hermitian matrix $A = iB$ we have

$$\hat{f}(\boldsymbol{k}) = \Bigl(\frac{\kappa}{\sqrt{2\pi}}\Bigr)^n |\det B|^{-\frac{1}{2}}e^{-i\sigma(B)\pi/4}e^{i\frac{1}{2}B^{-1}\boldsymbol{k},\cdot\boldsymbol{k}}. \tag{5.2.A.6}$$

where $\sigma(B) = \sigma_+(B) - \sigma_-(B)$, $\sigma_\pm(B)$ is the number of positive (negative) eigenvalues of B.

Remark 5.2.A.3. Poisson summation formula Theorem 5.2.5is replaced by

$$\sum_{\boldsymbol{m}\in\Gamma} f(\boldsymbol{m}) = \sum_{\boldsymbol{k}\in\Gamma^*} (2\pi)^n|\Omega|^{-1}\hat{f}(\boldsymbol{k}); \tag{5.2.A.7}$$

see Section 4.5.6.

Appendix 5.2.B. Fourier transform in the complex domain

Introduction

In this Appendix the familiarity with elements of the Complex Variables (like MAT334 at University of Toronto) is assumed.

When we can take in the definition

$$\hat{u}(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x)e^{-ixz}\,dx \tag{5.2.B.1}$$

complex z?

(a) Observe first that if

$$|u(x)| \le C\begin{cases} e^{-ax} & x\le 0,\\ e^{-bx} & x\ge 0,\end{cases} \tag{5.2.B.2}$$

with $a < b$, then integral (5.2.B.1) converges in the strip $\{z\colon a < \operatorname{Im}(z) < b\}$ and defines here a holomorphic function, which also tends to 0 as $|z|\to\infty$, $a+\epsilon \le \operatorname{Im}(z) \le b-\epsilon$ with arbitrary $\epsilon > 0$.

(b) In particular, if

$$u(x) = 0 \qquad \text{for } \ x\le 0 \tag{5.2.B.3}$$

and satisfies (5.2.B.2), then integral (5.2.B.1) converges in the lower half-plane $\{z\colon \operatorname{Im}(z) < a\}$ and defines here a holomorphic function, which also tends to 0 as $|z|\to\infty$, $\operatorname{Im}(z)\le a-\epsilon$ with arbitrary $\epsilon > 0$.

(c) On the other hand, (almost) converse is also true due to Paley-Wiener theorem.

Paley-Wiener Theorem

Theorem 5.2.B.1 (Paley-Wiener theorem). *The following statements are equivalent:*

(i) $f(z)$ *is holomorphic in lower half-plane* $\{z\colon \operatorname{Im}(z) > 0\}$,

$$\int_{-\infty}^{\infty} |f(\xi - i\eta)|^2\,d\xi \le M \qquad \forall \eta\ge 0 \tag{5.2.B.4}$$

(ii) There exists a function u, $u(x) = 0$ for $x < 0$ and

$$\int_{-\infty}^{\infty} |u(x)|^2 \, dx \le M \tag{5.2.B.5}$$

such that $f(z) = \hat{u}(z)$ for $z\colon \operatorname{Im}(z) \le 0$.

Remark 5.2.B.1. To consider functions holomorphic in in the lower half-plane $\{z\colon \operatorname{Im}(z) < a\}$ one needs to apply Paley-Wiener theorem to $g(z) = f(z - ia)$ with $c < a$.

To prove Paley-Wiener theorem one needs to consider Fourier integral

$$\begin{aligned} \hat{u}(z) = u(x) &:= \int_{-\infty}^{\infty} f(\xi) e^{ix\xi} \, d\xi \\ &= \int_{-\infty + i\eta}^{\infty + i\eta} f(z) e^{ixz} \, dz \end{aligned}$$

where we changed the contour of integration (and one can prove that the integral has not changed) observe, that for $x < 0$ this integral tends to 0 as $\eta \to +\infty$.

Laplace transform

Laplace transform is defined for functions $u : [0, \infty) \to \mathbb{C}$ such that

$$|u(x)| \le C e^{ax} \tag{5.2.B.6}$$

by

$$\mathcal{L}[u](p) = \int_0^{\infty} e^{-px} u(x) \, dx, \qquad \operatorname{Re}(p) > a. \tag{5.2.B.7}$$

Obviously, it could be described this way: extend $u(x)$ by 0 to $(-\infty, 0)$, then make a Fourier transform (5.2.B.1), and replace $z = -ip$; then $\operatorname{Im}(z) < a$ translates into $\operatorname{Re}(p) > a$.

Properties of Fourier transform translate into properties Laplace transform, but with a twist

$$(f * g)(x) := \int_0^x f(y) g(x - y) \, dy, \tag{5.2.B.8}$$

$$\mathcal{L}[u'](p) = p\mathcal{L}[u](p) - pu(0^+). \tag{5.2.B.9}$$

One can prove (5.2.B.9) by integration by parts. Those who are familiar with distributions (see Section 11.1 can obtain it directly because

$$(\mathcal{E}(u))'(x) = \mathcal{E}(u')(x) + u^+(0)\delta(x), \tag{5.2.B.10}$$

where $\mathcal{E}$ is an operator of extension by 0 from $[0,\infty)$ to $(-\infty,\infty)$ and δ is a Dirac delta function.

The Laplace transform provides a foundation to Operational Calculus by Oliver Heaviside. Its applications to Ordinary Differential Equations could be found in Chapter 6 of Boyce-DiPrima textbook.

Using complex variables

Complex variables could be useful to find Fourier and inverse Fourier transforms of certain functions.

Example 5.2.B.1. Let us find Fourier transform of $f(x) = \dfrac{1}{x^2+a^2}$, $\operatorname{Re}(a) > 0$.

$$\hat{f}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-ikx}\,dx}{x^2+a^2}.$$

As $k \gtrless 0$ function $\frac{e^{-ikz}}{z^2+a^2}$ is holomorphic at $\{z\colon \operatorname{Im}(z) \lessgtr 0\}$ except $z = \mp ia$, and nicely decays; then

$$\hat{f}(k) = \mp i \operatorname{Res}\Bigl(\frac{e^{-ikz}}{z^2+a^2}; z = \mp ia\Bigr) = \mp i\frac{e^{-ikz}}{2z}\Bigr|_{z=\pm ia} = \frac{1}{2}e^{-|k|}.$$

One can apply the same arguments to any rational function $\dfrac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials of order m and n, $m < n$ and $Q(x)$ does not have real roots.

Appendix 5.2.C. Discrete Fourier transform

Sometimes decomposition into complex Fourir series is called discrete Fourier transform. Namely, consider L-periodic function (assuming for simplicity that it is sufficiently smooth) $f(x)$. Then it can be decomposed into Fourier series, which, following Section 5.1 we write as

$$f(x) = \sum_{k_n} C(k_n) e^{k_n x} \Delta k_n, \tag{5.2.C.1}$$

$$C(k) := \frac{1}{2\pi} \int_I e^{-ik_n x}\, dx \tag{5.2.C.2}$$

with $k_n = \frac{2\pi i n}{L}$, $h = \Delta k_k = \frac{2\pi}{L}$. Now k is a discrete variable, running lattice (grid) $K := \frac{2\pi}{L}\mathbb{Z}$.

Let us call $C(k)$ *discrete Fourier transform.*

Which properties of Fourier transform (see Section 5.2) hold for discrete Fourier transform, and which should be modified?

From Theorem 5.2.3 survive Statements (i), (ii) (with $b \in K$ now, so periodicity does not break), (iii) and (v) (but only with $\lambda \in \mathbb{Z}$, so periodicity does not break).

Statement (iv) obviously does not, since multiplication by x breaks periodicity. Instead consider multiplications by $\frac{1}{hi}\bigl(e^{ixh} - 1\bigr)$, $\frac{1}{hi}\bigl(1 - e^{-ixh}\bigr)$ and $\frac{1}{h}\sin(xh)$. These multiplications become *finite differences*

$$\begin{aligned}
\Lambda_h^- \phi(k) &= \frac{1}{h}\bigl(\phi(k) - \phi(k-h)\bigr),\\
\Lambda_h^+ \phi(k) &= \frac{1}{h}\bigl(\phi(k+h) - \phi(k)\bigr),\\
\Lambda_h^- \phi(k) &= \frac{1}{2h}\bigl(\phi(k+h) - \phi(k-h)\bigr)
\end{aligned}$$

while multiplication by $\frac{2}{h^2}(\cos(xh) - 1)$ becomes a *discrete Laplacian*

$$\Delta_h \phi(k) = \frac{1}{h^2}\bigl(\phi(k+h) - 2\phi(k) + \phi(k-h)\bigr).$$

From Theorem 5.2.5 Statement (i) does not survive since convolution of periodic functions is not necessarily defined, but Statement (ii) survives with a *discrete convolution*

$$(f * g)(k) = \sum_{\omega \in K} f(\omega) g(k - \omega) h.$$

Those who are familiar with distributions (see Section 11.1) can observe that for L-periodic functions ordinary Fourier transform is

$$\hat{f}(\omega) = \sum_{k \in K} C(k) \delta(\omega - k) h$$

where δ is a Dirac delta-function.

Problems to Sections 5.1 and 5.2

Some of the problems could be solved based on the other problems and properties of Fourier transform (see Section 5.2) and such solutions are much shorter than from the scratch; seeing and exploiting connections is a plus.

Problem 1. Let F be an operator of Fourier transform: $f(x) \to \hat{f}(k)$. Prove that

(a) $F^*F = FF^* = I$;

(b) $(F^2 f)(x) = f(-x)$ and therefore $F^2 f = f$ for even function f and $F^2 = -f$ for odd function f;

(c) $F^4 = I$;

(d) If f is a real-valued function then $\hat{f}$ is real-valued if and only if f is even and $i\hat{f}$ is real-valued if and only if f is odd.

Problem 2. Let $\alpha > 0$. Find Fourier transforms of

(a) $f(x) = e^{-\alpha|x|}$;

(b) $f(x) = e^{-\alpha|x|}\cos(\beta x)$;

(c) $f(x) = e^{-\alpha|x|}\sin(\beta x)$;

(d) $f(x) = xe^{-\alpha|x|}\sin(\beta x)$;

(e) $f(x) = \begin{cases} e^{-\alpha x} & x > 0, \\ 0 & x \le 0; \end{cases}$

(f) $f(x) = xe^{-\alpha|x|}$;

(g) $f(x) = xe^{-\alpha|x|}\cos(\beta x)$;

(h) $f(x) = x^2 e^{-\alpha|x|}$.

(i) $f(x) = \begin{cases} xe^{-\alpha x} & x > 0, \\ 0 & x \le 0; \end{cases}$

Problem 3. Let $\alpha > 0$. Find Fourier transforms of

(a) $(x^2 + \alpha^2)^{-1}$;

(b) $x(x^2 + \alpha^2)^{-1}$;

(c) $(x^2 + \alpha^2)^{-1}\cos(\beta x)$,

(d) $(x^2 + \alpha^2)^{-1}\sin(\beta x)$;

(e) $x(x^2 + \alpha^2)^{-1}\cos(\beta x)$,

(f) $x(x^2 + \alpha^2)^{-1}\sin(\beta x)$.

Problem 4. Let $\alpha > 0$. Based on Fourier transform of $e^{-\alpha x^2/2}$ find Fourier transforms of

(a) $f(x) = xe^{-\alpha x^2/2}$;

(b) $f(x) = x^2 e^{-\alpha x^2/2}$;

(c) $e^{-\alpha x^2/2}\cos(\beta x)$,

(d) $e^{-\alpha x^2/2}\sin(\beta x)$;

(e) $xe^{-\alpha x^2/2}\cos(\beta x)$,

(f) $xe^{-\alpha x^2/2}\sin(\beta x)$.

Problem 5. Find Fourier transforms of
Let $a > 0$. Find Fourier transforms $\hat{f}(k)$ of functions

(a) $f(x) = \begin{cases} 1 & |x| \le a, \\ 0 & |x| > a; \end{cases}$

(b) $f(x) = \begin{cases} x & |x| \le a, \\ 0 & |x| > a; \end{cases}$

(c) $f(x) = \begin{cases} |x| & |x| \le a, \\ 0 & |x| > a; \end{cases}$

(d) $f(x) = \begin{cases} a - |x| & |x| \le a, \\ 0 & |x| > a; \end{cases}$

(e) $f(x) = \begin{cases} a^2 - x^2 & |x| \le a, \\ 0 & |x| > a; \end{cases}$

(f) Using (a) calculate $\int_{-\infty}^{\infty} \frac{\sin(x)}{x}\,dx$.

Problem 6. Using Complex Variables class (if you took one) find directly Fourier transforms $\hat{f}(k)$ of functions

(a) $(x^2 + a^2)^{-1}$ with $a > 0$;

(b) $(x^2+a^2)^{-1}(x^2+b^2)^{-1}$, with $a > 0$, $b > 0$, $b \ne a$;

(c) $(x^2 + a^2)^{-2}$ with $a > 0$;

(d) $x(x^2 + a^2)^{-2}$ with $a > 0$;

(e) $(x^2+a^2)^{-1}(x^2+b^2)^{-1}$, with $a > 0$, $b > 0$, $b \ne a$;

(f) $x(x^2+a^2)^{-1}(x^2+b^2)^{-1}$, with $a > 0$, $b > 0$, $b \ne a$.

Problem 7. (a) Prove the same properties as in 1 for multidimensional Fourier tramsform (see Subsection 5.2.5.

(b) Prove that f if multidimensional function f has a rotational symmetry (that means $f(Q\boldsymbol{x}) = f(\boldsymbol{x})$ for all orthogonal transform Q) then $\hat{f}$ also has a rotational symmetry (and conversely).

Note. Equivalently f has a rotational symmetry if $f(\boldsymbol{x})$ depend only on $|\boldsymbol{x}|$.

Problem 8. Find multidimenxional Fourier transforms of

(a) $f(\boldsymbol{x}) = \begin{cases} 1 & |\boldsymbol{x}| \le a, \\ 0 & |\boldsymbol{x}| > a; \end{cases}$

(b) $f(\boldsymbol{x}) = \begin{cases} a - |\boldsymbol{x}| & |\boldsymbol{x}| \le a, \\ 0 & |\boldsymbol{x}| > a; \end{cases}$

(c) $f(\boldsymbol{x}) = \begin{cases} (a - |\boldsymbol{x}|)^2 & |\boldsymbol{x}| \le a, \\ 0 & |\boldsymbol{x}| > a; \end{cases}$

(d) $f(\boldsymbol{x}) = \begin{cases} a^2 - |\boldsymbol{x}|^2 & |\boldsymbol{x}| \le a, \\ 0 & |\boldsymbol{x}| > a. \end{cases}$

(e) $f(x) = e^{-\alpha|\boldsymbol{x}|}$;

(f) $f(\boldsymbol{x}) = |\boldsymbol{x}|e^{-\alpha|\boldsymbol{x}|}$;

(g) $f(\boldsymbol{x}) = |\boldsymbol{x}|^2 e^{-\alpha|\boldsymbol{x}|}$.

Hint. Using Problem 7(b) observe that we need to find only $\hat{f}(0, \dots, 0, k)$ and use appropriate coordinate system (polar as $n = 2$, or spherical as $n = 3$ and so one).

Note. This problem could be solved as $n = 2$, $n = 3$ or $n \ge 2$ (any).

5.3 Applications of Fourier transform to PDEs

5.3.1 Heat Eequation

Consider problem

$$u_t = k u_{xx}, \qquad t > 0,\ -\infty < x < \infty, \tag{5.3.1}$$

$$u|_{t=0} = g(x). \tag{5.3.2}$$

Making partial Fourier transform with respect to $x \mapsto \xi$ (so $u(x,t) \mapsto \hat{u}(\xi,t)$) we arrive to

$$\hat{u}_t = -k\xi^2 \hat{u}, \tag{5.3.3}$$

$$\hat{u}|_{t=0} = \hat{g}(\xi). \tag{5.3.4}$$

Indeed, $\partial_x \mapsto i\xi$ and therefore $\partial_x^2 \mapsto -\xi^2$.

Note that (5.3.3) (which is $\hat{u}_t = -k\xi^2\hat{u}$) is an ODE and solving it we arrive to $\hat{u} = A(\xi)e^{-k\xi^2 t}$; plugging into (5.3.4) (which is $\hat{u}|_{t=0} = \hat{g}(\xi)$) we find that $A(\xi) = \hat{g}(\xi)$ and therefore

$$\hat{u}(\xi, t) = \hat{g}(\xi)e^{-k\xi^2 t}. \tag{5.3.5}$$

The right-hand expression is a product of two Fourier transforms, one is $\hat{g}(\xi)$ and another is Fourier transform of IFT of $e^{-k\xi^2 t}$ (reverse engineering?).

If we had $e^{-\xi^2/2}$ we would have IFT equal to $\sqrt{2\pi}e^{-x^2/2}$; but we can get from $e^{-\xi^2/2}$ to $e^{-k\xi^2 t}$ by scaling $\xi \mapsto (2kt)^{\frac{1}{2}}\xi$ and therefore $x \mapsto (2kt)^{-\frac{1}{2}}x$ (and we need to multiply the result by by $(2kt)^{-\frac{1}{2}}$);

therefore $e^{-k\xi^2 t}$ is a Fourier transform of $\frac{\sqrt{2\pi}}{\sqrt{2kt}}e^{-x^2/4kt}$.

Again: $\hat{u}(\xi,t)$ is a product of FT of g and of $\frac{\sqrt{2\pi}}{\sqrt{2kt}}e^{-x^2/4kt}$ and therefore u is the convolution of these functions (multiplied by $1/(2\pi)$):

$$u(x,t) = g * \frac{1}{\sqrt{4\pi kt}}e^{-\frac{x^2}{4kt}} = \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} g(x')e^{-\frac{(x-x')^2}{4kt}}\,dx'. \tag{5.3.6}$$

We recovered formula which we had already.

Remark 5.3.1. Formula (5.3.5) shows that the problem is really ill-posed for $t < 0$.

5.3.2 Schrödinger equation

Consider problem

$$u_t = iku_{xx}, \qquad t > 0,\ -\infty < x < \infty, \tag{5.3.7}$$

$$u|_{t=0} = g(x). \tag{5.3.8}$$

Making partial Fourier transform with respect to $x \mapsto \xi$ (so $u(x,t) \mapsto \hat{u}(\xi,t)$) we arrive to

$$\hat{u}_t = -ik\xi^2\hat{u}, \tag{5.3.9}$$

$$\hat{u}|_{t=0} = \hat{g}(\xi). \tag{5.3.10}$$

Note that (5.3.9) is an ODE and solving it we arrive to $\hat{u} = A(\xi)e^{-ik\xi^2 t}$; plugging into (5.3.10) we find that $A(\xi) = \hat{g}(\xi)$ and therefore

$$\hat{u}(\xi,t) = \hat{g}(\xi)e^{-ik\xi^2 t}. \tag{5.3.11}$$

The right-hand expression is a product of two Fourier transforms, one is $\hat{g}(\xi)$ and another is Fourier transform of IFT of $e^{-ik\xi^2 t}$ (reverse engineering?).

As it was explained in Section 5.2 that we need just to plug ik instead of k (as $t > 0$) into the formulae we got before; so instead of $\frac{1}{\sqrt{4\pi kt}}e^{-\frac{x^2}{4kt}}$ we get $\frac{1}{\sqrt{4\pi kit}}e^{\frac{x^2}{4kit}} = \frac{1}{\sqrt{4\pi kt}}e^{-\frac{\pi i}{4}+\frac{ix^2}{4kt}}$ because we need to take a correct branch of $\sqrt{i} = e^{\frac{i\pi}{4}}$.

As $t < 0$ we need to replace t by $-t$ and i by $-i$ resulting in $\frac{1}{\sqrt{4\pi k|t|}}e^{\frac{\pi i}{4}+\frac{ix^2}{4kt}}$.

Therefore

$$u(x,t) = \frac{1}{\sqrt{4\pi k|t|}}\int_{-\infty}^{\infty} g(x')e^{\mp\frac{i\pi}{4}+\frac{i(x-x')^2}{4kt}}\,dx' \tag{5.3.12}$$

as $\pm t > 0$.

Remark 5.3.2. Formula (5.3.11) shows that the problem is well-posed for both $t > 0$ and $t < 0$.

5.3.3 Laplace equation in half-plane

Consider problem

$$\Delta u := u_{xx} + u_{yy} = 0, \qquad y > 0,\ -\infty < x < \infty, \tag{5.3.13}$$

$$u|_{y=0} = g(x). \tag{5.3.14}$$

Remark 5.3.3. This problem definitely is not uniquely solvable (f.e. $u = y$ satisfies homogeneous boundary condition) and to make it uniquely solvable we need to add condition $|u| \le M$.

Making partial Fourier transform with respect to $x \mapsto \xi$ (so $u(x,t) \mapsto \hat{u}(\xi,t)$) we arrive to

$$\hat{u}_{yy} - \xi^2\hat{u} = 0, \tag{5.3.15}$$

$$\hat{u}|_{y=0} = \hat{g}(\xi). \tag{5.3.16}$$

Note that (5.3.15) (which is $\hat{u}_{yy} - \xi^2\hat{u} = 0$) is an ODE and solving it we arrive to

$$\hat{u}(\xi,y) = A(\xi)e^{-|\xi|y} + B(\xi)e^{|\xi|y}. \tag{5.3.17}$$

Indeed, characteristic equation $\alpha^2 - \xi^2$ has two roots $\alpha_{1,2} = \pm|\xi|$; *we take $\pm|\xi|$ instead of just $\pm\xi$ because we need to control signs.*

We discard the second term in the right-hand expression of (5.3.17) *because it is unbounded.* However if we had Cauchy problem (i.e. $u|_{y=0} = g(x)$, $u_y|_{y=0} = h(x)$) we would not be able to do this and this problem will be ill-posed.

So, $\hat{u} = A(\xi)e^{-|\xi|y}$ and (5.3.16) yields $A(\xi) = \hat{g}(\xi)$:

$$\hat{u}(\xi, y) = \hat{g}(\xi)e^{-|\xi|y}. \tag{5.3.18}$$

Now we need to find the IFT of $e^{-|\xi|y}$. This calculations are easy (do them!) and IFT is $2y(x^2 + y^2)^{-1}$. Then

$$u(x, y) = \frac{1}{\pi}\int_{-\infty}^{\infty} g(x')\frac{y}{(x - x')^2 + y^2}\, dx'. \tag{5.3.19}$$

Remark 5.3.4. Setting $y = 0$ we see that $u|_{y=0} = 0$. Contradiction?–No, we cannot just set $y = 0$. We need to find a limit as $y \to +0$, and note that $\dfrac{y}{(x - x')^2 + y^2} \to 0$ except as $x' = x$ and

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{(x - x')^2 + y^2}\, dx' = 1$$

so the limit will be $g(x)$ as it should be.

5.3.4 Laplace equation in half-plane. II

Replace Dirichlet boundary condition by Robin boundary condition

$$\Delta u := u_{xx} + u_{yy} = 0, \qquad y > 0,\ -\infty < x < \infty, \tag{5.3.13}$$

$$(u_y - \alpha u)|_{y=0} = h(x). \tag{5.3.20}$$

Then (5.3.16) (which is $\hat{u}|_{y=0} = \hat{g}(\xi)$) should be replaced by

$$(\hat{u}_y - \alpha\hat{u})|_{y=0} = \hat{h}(\xi). \tag{5.3.21}$$

and then

$$A(\xi) = -(|\xi| + \alpha)^{-1}\hat{h}(\xi) \tag{5.3.22}$$

and

$$\hat{u}(\xi, y) = -\hat{h}(\xi)(|\xi| + \alpha)^{-1}e^{-|\xi|y}. \tag{5.3.23}$$

The right-hand expression is a nice function provided $\alpha > 0$ (and this is correct from the physical point of view) and therefore everything is fine (but we just cannot calculate explicitly IFT of $(|\xi| + \alpha)^{-1}e^{-|\xi|y}$).

Consider Neumann boundary condition i.e. set $\alpha = 0$. Then we have a trouble: $-\hat{h}(\xi)(|\xi| + \alpha)^{-1}e^{-|\xi|y}$ could be singular at $\xi = 0$ and to avoid it we assume that $\hat{h}(0) = 0$. This means exactly that

$$\int_{-\infty}^{\infty} h(x)\,dx = 0 \tag{5.3.24}$$

and this condition we really need and it is justified from the physical point of view: f.e. if we are looking for stationary heat distribution and we have heat flow defined, we need to assume that the total flow is 0 (otherwise the will be no stationary distribution!).

So we need to calculate IFT of $-|\xi|^{-1}e^{-|\xi|y}$. Note that derivative of this with respect to y is $e^{-|\xi|y}$ which has an IFT $\dfrac{y}{\pi(x^2+y^2)}$; integrating with respect to y we get $\frac{1}{2\pi}\log(x^2+y^2) + c$ and therefore

$$u(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} h(x')\log\bigl((x-x')^2+y^2\bigr)\,dx' + C. \tag{5.3.25}$$

Remark 5.3.5. (a) Here C is an arbitrary constant. Again, the same physical interpretation: knowing heat flow we define solution up to a constant as the total heat energy is arbitrary.

(b) Formula (5.3.25) gives us a solution which can grow as $|x| \to \infty$ even if h is fast decaying there (or even if $h(x) = 0$ as $|x| \ge c$).

However as $|x| \gg 1$ and h is fast decaying $\bigl((x-x')^2+y^2\bigr) \approx \bigl(x^2+y^2\bigr)$ (with a small error) and growing part of u is $\frac{1}{2\pi}\log\bigl(x^2+y^2\bigr)\int_{-\infty}^{\infty} h(x')\,dx'$ which is 0 precisely because of condition (5.3.24): $\int_{-\infty}^{\infty} h(x')\,dx' = 0$.

5.3.5 Laplace equation in strip

Consider problem

$$\Delta u := u_{xx} + u_{yy} = 0, \qquad 0 < y < b,\ -\infty < x < \infty, \tag{5.3.26}$$

$$u|_{y=0} = g(x), \tag{5.3.27}$$

$$u|_{y=b} = h(x). \tag{5.3.28}$$

Then we get (5.3.17) again

$$\hat{u}(\xi,y) = A(\xi)e^{-|\xi|y} + B(\xi)e^{|\xi|y} \tag{5.3.17}$$

but with two boundary condition we cannot discard anything; we get instead

$$A(\xi) \qquad + B(\xi) \qquad = \hat{g}(\xi), \tag{5.3.29}$$

$$A(\xi)e^{-|\xi|b} + B(\xi)e^{|\xi|b} = \hat{h}(\xi) \tag{5.3.30}$$

which implies

$$A(\xi) = \frac{e^{|\xi|b}}{2\sinh(|\xi|b)}\hat{g}(\xi) - \frac{1}{2\sinh(|\xi|b)}\hat{h}(\xi),$$

$$B(\xi) = -\frac{e^{-|\xi|b}}{2\sinh(|\xi|b)}\hat{g}(\xi) + \frac{1}{2\sinh(|\xi|b)}\hat{h}(\xi)$$

and therefore

$$\hat{u}(\xi, y) = \frac{\sinh(|\xi|(b-y))}{\sinh(|\xi|b)}\hat{g}(\xi) + \frac{\sinh(|\xi|y)}{\sinh(|\xi|b)}\hat{h}(\xi). \tag{5.3.31}$$

One can see easily that $\dfrac{\sinh(|\xi|(b-y))}{\sinh(|\xi|b)}$ and $\dfrac{\sinh(|\xi|y)}{\sinh(|\xi|b)}$ are bounded for $0 \le y \le b$ and fast decaying as $|\xi| \to \infty$ for $y \ge \epsilon$ (for $y \le b - \epsilon$ respectively) with arbitrarily small $\epsilon > 0$.

Problem 5.3.1. Investigate other boundary conditions (Robin, Neumann, mixed–Dirichlet at $y = 0$ and Neumann at $y = b$ and so on.).

5.3.6 1D Wave equation

Consider problem

$$u_{tt} = c^2 u_{xx}, \qquad -\infty < x < \infty, \tag{5.3.32}$$

$$u|_{t=0} = g(x), \tag{5.3.33}$$

$$u_t|_{t=0} = h(x). \tag{5.3.34}$$

Making partial Fourier transform with respect to $x \mapsto \xi$ we arrive to

$$\hat{u}_{tt} = -c^2\xi^2 \hat{u}_{xx}, \tag{5.3.35}$$

$$\hat{u}|_{t=0} = \hat{g}(\xi), \tag{5.3.36}$$

$$\hat{u}_t|_{t=0} = \hat{h}(\xi). \tag{5.3.37}$$

Then characteristic equation for ODE (5.3.35) (which is $\hat{u}_{tt} = -c^2\xi^2\hat{u}_{xx}$) is $\alpha^2 = -c^2\xi^2$ and $\alpha_{1,2} = \pm ic\xi$,

$$\hat{u}(\xi,t) = A(\xi)\cos(c\xi t) + B(\xi)\sin(c\xi t)$$

with initial conditions implying $A(\xi) = \hat{g}(\xi)$, $B(\xi) = 1/(ci\xi)\cdot\hat{h}(\xi)$ and

$$\hat{u}(\xi,t) = \hat{g}(\xi)\cos(c\xi t) + \hat{h}(\xi)\cdot\frac{1}{c\xi}\sin(c\xi t). \tag{5.3.38}$$

Rewriting $\cos(c\xi t) = \frac{1}{2}\bigl(e^{ic\xi t} + e^{-ic\xi t}\bigr)$ and recalling that multiplication of FT by $e^{i\xi b}$ is equivalent to the shifting original to the left by b we conclude that $\hat{g}(\xi)\cos(c\xi t)$ *is a Fourier transform of* $\frac{1}{2}\bigl(g(x+ct) + g(x-ct)\bigr)$.

If we denote H as a primitive of h then $\hat{h}(\xi)\cdot\frac{1}{c\xi}\sin(c\xi t) = \hat{H}(\xi)\cdot\frac{1}{c}i\sin(c\xi t)$ which in virtue of the same arguments is *FT of* $\frac{1}{2c}\bigl(H(x+ct) - H(x-ct)\bigr) = \frac{1}{2c}\int_{x-ct}^{x+ct} h(x')\,dx'$.

Therefore

$$u(x,t) = \frac{1}{2}\bigl(g(x+ct) + g(x-ct)\bigr) + \frac{1}{2c}\int_{x-ct}^{x+ct} h(x')\,dx' \tag{5.3.39}$$

and we arrive again to D'Alembert formula.

5.3.7 Airy equation

Airy equation is 2-nd order ODE

$$y'' - xy = 0$$

which plays an important role in the theory of electromagnetic wave propagation (toy-model for caustics and convex-obstacle diffraction). Making Fourier transform we arrive to

$$-\xi^2\hat{y} + i\hat{y}' = 0$$

which implies

$$\hat{y} = Ce^{\frac{1}{3}i\xi^3} \implies y(x) = C\int_{-\infty}^{\infty} e^{\frac{1}{3}i\xi^3 + ix\xi}d\xi.$$

This is *Airy function* (when studying it, Complex Variables are very handy).

One can ask, why we got only one linearly independent solution? 2-nd order ODE must have 2. The second solution, however, grows really fast (superexponentially) as $x \to +\infty$ and is cut-off by Fourier transform.

5.3.8 Multidimensional equations

Multidimensional equations are treated in the same way:

Heat and Schrödinger equations. We make partial FT (with respect to spatial variables) and we get

$$\begin{aligned} \hat{u}(\boldsymbol{\xi}, t) &= \hat{g}(\boldsymbol{\xi}) e^{-k|\xi|^2 t}, \\ \hat{u}(\boldsymbol{\xi}, t) &= \hat{g}(\boldsymbol{\xi}) e^{-ik|\xi|^2 t} \end{aligned}$$

respectively where $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n)$, $|\boldsymbol{\xi}| = (\xi_1^2 + \ldots \xi_n^2)^{\frac{1}{2}}$; in this case $e^{-k|\boldsymbol{\xi}|^2 t} = \prod_{j=1}^n e^{-k|\xi_j|^2 t}$ is a product of functions depending on different variables, so IFT will be again such product and we have IFT equal

$$\prod_{j=1}^{n} \frac{1}{\sqrt{4\pi kt}} e^{-|x_j|^2/4kt} = (4\pi kt)^{-\frac{n}{2}} e^{-|\boldsymbol{x}|^2/4kt}$$

for heat equation and similarly for Schrödinger equation and we get a solution as a multidimensional convolution. Here $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $|\boldsymbol{x}| = (x_1^2 + \ldots + x_n^2)^{\frac{1}{2}}$.

Wave equation. We do the same but now

$$\hat{u}(\boldsymbol{\xi}, t) = \hat{g}(\boldsymbol{\xi}) \cos(c|\boldsymbol{\xi}|t) + \hat{h}(\boldsymbol{\xi}) \cdot \frac{1}{c|\boldsymbol{\xi}|} \sin(c|\boldsymbol{\xi}|t).$$

Finding IFT is not easy. We will do it (kind of) for $n = 3$ in Section 9.1.

Laplace equation. We consider it in $\mathbb{R}^n \times I \ni (\boldsymbol{x}; y)$ with either $I = \{y : y > 0\}$ or $I = \{y : 0 < y < b\}$ and again make partial FT with respect to $\boldsymbol{x}$ but not y.

Problems to Section 5.3

Problem 1. (a) Consider Dirichlet problem

$$\begin{aligned} &u_{xx} + u_{yy} = 0, \qquad -\infty < x < \infty, y > 0, \\ &u|_{y=0} = f(x). \end{aligned}$$

Make Fourier transform by x, solve problem for ODE for $\hat{u}(k, y)$ which you get as a result and write $u(x, y)$ as a Fourier integral.

(b) Consider Neumann problem

$$u_{xx} + u_{yy} = 0, \qquad -\infty < x < \infty, y > 0,$$
$$u_y|_{y=0} = f(x).$$

Make Fourier transform by x, solve problem for ODE for $\hat{u}(k, y)$ which you get as a result and write $u(x, y)$ as a Fourier integral. What condition must satisfy f?

Problem 2. (a) Consider Dirichlet problem

$$u_{xx} + u_{yy} = 0, \qquad -\infty < x < \infty, 0 < y < 1,$$
$$u|_{y=0} = f(x), \quad u|_{y=1} = g(x).$$

Make Fourier transform by x, solve problem for ODE for $\hat{u}(k, y)$ which you get as a result and write $u(x, y)$ as a Fourier integral.

(b) Consider Dirichlet-Neumann problem

$$u_{xx} + u_{yy} = 0, \qquad -\infty < x < \infty, 0 < y < 1,$$
$$u|_{y=0} = f(x), \quad u_y|_{y=1} = g(x).$$

Make Fourier transform by x, solve problem for ODE for $\hat{u}(k, y)$ which you get as a result and write $u(x, y)$ as a Fourier integral.

(c) Consider Neumann problem

$$u_{xx} + u_{yy} = 0, \qquad -\infty < x < \infty, 0 < y < 1,$$
$$u_y|_{y=0} = f(x), \quad u_y|_{y=1} = g(x).$$

Make Fourier transform by x, solve problem for ODE for $\hat{u}(k, y)$ which you get as a result and write $u(x, y)$ as a Fourier integral. What condition must satisfy f, g?

Problem 3. Consider Robin problem

$$u_{xx} + u_{yy} = 0, \qquad -\infty < x < \infty, y > 0,$$
$$(u_y + \alpha u)|_{y=0} = f(x).$$

Make Fourier transform by x, solve problem for ODE for $\hat{u}(k, y)$ which you get as a result and write $u(x, y)$ as a Fourier integral. What condition (if any) must satisfy f?

Hint. Consider separately cases

(A) $\alpha \in \mathbb{C} \setminus [0, \infty)$ and

(B) $\alpha \in [0, \infty)$.

Problem 4. (a) Consider problem

$$\begin{aligned} &\Delta^2 = 0, \qquad -\infty < x < \infty, y > 0,\\ &u|_{y=0} = f(x), \qquad\qquad u_y|_{y=0} = g(x). \end{aligned}$$

Make Fourier transform by x, solve problem for ODE for $\hat{u}(k, y)$ which you get as a result and write $u(x, y)$ as a Fourier integral.

(b) Consider problem

$$\begin{aligned} &\Delta^2 = 0, \qquad -\infty < x < \infty, y > 0,\\ &u_{yy}|_{y=0} = f(x), \qquad\qquad \Delta u_y|_{y=0} = g(x). \end{aligned}$$

Make Fourier transform by x, solve problem for ODE for $\hat{u}(k, y)$ which you get as a result and write $u(x, y)$ as a Fourier integral. What condition must satisfy f, g?

Problem 5. In the problems below solution $u(x, t)$, must be represented in the form of the appropriate Fourier integral:

$$\text{(a)} \quad \begin{cases} u_{tt} = -4u_{xxxx} & -\infty < x < \infty,\ -\infty < t < \infty,\\ u|_{t=0} = \begin{cases} 1 & |x| < 2,\\ 0 & |x| \geq 2, \end{cases} & u_t|_{t=0} = 0,\\ \max |u| < \infty. \end{cases}$$

$$\text{(b)} \quad \begin{cases} u_t = 4u_{xx} & -\infty < x < \infty,\ t > 0,\\ u|_{t=0} = e^{-|x|}\\ \max |u| < \infty. \end{cases}$$

$$\text{(c)} \quad \begin{cases} u_t = u_{xx} & -\infty < x < \infty,\ t > 0,\\ u|_{t=0} = \begin{cases} 1 - x^2 & |x| < 1,\\ 0 & |x| \geq 1, \end{cases}\\ \max |u| < \infty. \end{cases}$$

(d) $\begin{cases} u_{tt} = u_{xx} - 4u & -\infty < x < \infty, \\ u|_{t=0} = 0, & u_t|_{t=0} = e^{-x^2/2}. \end{cases}$

Problem 6. In the half-plane $\{(x,y)\colon x>0,\ -\infty<y<\infty\}$ find solution

$$\begin{aligned} &\Delta u = 0, \\ &u|_{x=0} = e^{-|y|}, \\ &\max |u| < \infty. \end{aligned}$$

Solution should be represented in the form of the appropriate Fourier integral.

Problem 7. In the strip $\{(x,y)\colon 0<x<1,\ -\infty<y<\infty\}$ find solution

$$\begin{aligned} &\Delta u - u = 0, \\ &u_x|_{x=0} = 0, \\ &u|_{x=1} = \begin{cases} \cos(y) & |y| \le \frac{\pi}{2}, \\ 0 & |y| \ge \frac{\pi}{2}, \end{cases} \\ &\max |u| < \infty. \end{aligned}$$

Solution should be represented in the form of the appropriate Fourier integral.

Problem 8. In the strip $\{(x,y)\colon 0<x<1,\ -\infty<y<\infty\}$ find solution

$$\begin{aligned} &\Delta u - 4u = 0, \\ &u_x|_{x=0} = 0, \\ &u_x|_{x=1} = \begin{cases} 1-|y| & |y| \le 1, \\ 0 & |y| \ge 1, \end{cases} \\ &\max |u| < \infty. \end{aligned}$$

Solution should be represented in the form of the appropriate Fourier integral.

Problem 9. In the strip $\{(x,y)\colon 0<x<1,\ -\infty<y<\infty\}$ find solution

$$\begin{aligned} &\Delta u = 0, \\ &u|_{x=0} = 0, \\ &u|_{x=1} = \begin{cases} 1-y^2 & |y| \le 1, \\ 0 & |y| \ge 1, \end{cases} \\ &\max |u| < \infty. \end{aligned}$$

Solution should be represented in the form of the appropriate Fourier integral.

Problem 10. Solve the following equations explicitly for $u(x)$ (that means do *not* leave your answer in integral form!).

(a) $e^{-\frac{x^2}{2}} = \frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-y|}u(y)\,dy$;

(b) $e^{-x^2} = \int_0^{\infty} f(x-y)e^{-y}dy$;

(c) $e^{-\frac{x^2}{4}} = \int_{-\infty}^{0} u(x-y)e^{\frac{1}{2}y}dy$;

(d) $e^{-\frac{x^2}{2}} = \frac{1}{\sqrt{6\pi}}\int_{-\infty}^{\infty} e^{-2(x-y)^2}u(y)\,dy$;

(e) $\frac{1}{4+x^2} = \int_{-\infty}^{\infty} \frac{u(x-y)}{y^2+1}\,dy$;

(f) $e^{-|x|} = \int_1^{\infty} e^{-y}u(x-y)dy$;

(g) $e^{-|x|} = \int_x^{\infty} e^{-2(y-x)}u(y)\,dy$;

(h) $xe^{-|x|} = \int_{-\infty}^{\infty} e^{-|x-y|}u(y)\,dy$.