(1) $$f(x) = a_0 + \sum_{n=1} (a_n \cos nx + b_n \sin nx).$$

Ve encountered these expansions in Section 1.2 when we discussed the vibrations of a plucked string. A ou will see in the following chapters, Fourier series are essential for the treatment of several othe nportant applications. To be able to use Fourier series, we need to know

> which functions have Fourier series expansions? and if a function has a Fourier series, how do we compute the coefficients $a_0, a_1, a_2, \ldots, b_1, b_2, \ldots$?

Ve will answer the second question in this section. As for the first question, its general treatment is wel eyond the level of this text. We will present conditions that are sufficient for functions to have a Fourie eries representation. These conditions are simple and general enough to cover all cases of interest to us Ve will focus on the applications and defer the proofs concerning the convergence of Fourier series t ections 2.8–2.10.

Euler Formulas for the Fourier Coefficients

'o derive the formulas for the coefficients that appear in (1), we proceed as Fourier himself did. W ntegrate both sides of (1) over the interval $[-\pi, \pi]$, assuming term-by-term integration is justified, and ge

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi} a_0\,dx + \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} (a_n \cos nx + b_n \sin nx)\;dx.$$

But because $\int_{-\pi}^{\pi} \cos nx\,dx = \int_{-\pi}^{\pi} \sin nx\,dx = 0$ for $n = 1, 2, \ldots$, it follows that

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi} a_0\,dx = 2\pi a_0,$$

r

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx.$$

Note that a_0 is the average of f on the interval $[-\pi, \pi]$. For the interpretation of the integral as an average ee Exercise 6, Section 2.5.) Similarly, starting with (1), we multiply both sides by $\cos mx$ ($m \geq 1$) ntegrate term-by-term, use the orthogonality of the trigonometric system (Section 2.1), and get

$$\int_{-\pi}^{\pi} f(x)\cos mx\,dx = \overbrace{\int_{-\pi}^{\pi} a_0 \cos mx\,dx}^{=0} + \sum_{n=1}^{\infty} \overbrace{\int_{-\pi}^{\pi} a_n \cos nx \cos mx\,dx}^{=0 \text{ for } m \neq n}$$

$$+ \sum_{n=1}^{\infty} \overbrace{\int_{-\pi}^{\pi} b_n \sin nx \cos mx\,dx}^{=0}$$

$$= a_m \overbrace{\int_{-\pi}^{\pi} \cos^2 mx\,dx}^{=\pi} = \pi a_m.$$

Hence

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx\,dx \quad (m = 1,\, 2,\, \ldots).$$

By a similar procedure,

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx\,dx \quad (m = 1,\, 2,\, \ldots).$$

The following box summarizes our discussion and contains basic definitions.

EULER FORMULAS FOR THE FOURIER COEFFICIENTS

Suppose that *f* has the Fourier series representation

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

Then the coefficients a_0, a_n, and b_n are called the Fourier coefficients of *f* and are given by the followi Euler formulas:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx, \tag{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \quad (n = 1, 2, \ldots), \tag{3}$$