We encountered these expansions in Section 1.2 when we discussed the vibrations of a plucked string. As you will see in the following chapters, Fourier series are essential for the treatment of several other important applications. To be able to use Fourier series, we need to know

which functions have Fourier series expansions? and if a function has a Fourier series, how do we compute the coefficients \(a_0, a_1, a_2, \ldots, b_1, b_2, \ldots\)?

We will answer the second question in this section. As for the first question, its general treatment is well beyond the level of this text. We will present conditions that are sufficient for functions to have a Fourier series representation. These conditions are simple and general enough to cover all cases of interest to us. We will focus on the applications and defer the proofs concerning the convergence of Fourier series to Sections 2.8–2.10.

**Euler Formulas for the Fourier Coefficients**

to derive the formulas for the coefficients that appear in (1), we proceed as Fourier himself did. We integrate both sides of (1) over the interval \([-\pi, \pi]\), assuming term-by-term integration is justified, and get

\[
\int_{-\pi}^{\pi} f(x) \, dx = \int_{-\pi}^{\pi} a_0 \, dx + \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} (a_n \cos nx + b_n \sin nx) \, dx.
\]

But because \(\int_{-\pi}^{\pi} \cos nx \, dx = \int_{-\pi}^{\pi} \sin nx \, dx = 0\) for \(n = 1, 2, \ldots\), it follows that

\[
\int_{-\pi}^{\pi} f(x) \, dx = \int_{-\pi}^{\pi} a_0 \, dx = 2\pi a_0,
\]

or

\[
a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx.
\]

Note that \(a_0\) is the average of \(f\) on the interval \([-\pi, \pi]\). For the interpretation of the integral as an average see Exercise 6, Section 2.5. Similarly, starting with (1), we multiply both sides by \(\cos mx\) \((m \geq 1)\), integrate term-by-term, use the orthogonality of the trigonometric system (Section 2.1), and get
\[
\int_{-\pi}^{\pi} f(x) \cos mx \, dx = \int_{-\pi}^{\pi} a_0 \cos mx \, dx + \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} a_n \cos nx \cos mx \, dx
\]

\[
= \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} b_n \sin nx \cos mx \, dx
\]

\[
= a_m \int_{-\pi}^{\pi} \cos^2 mx \, dx = \pi a_m.
\]

Hence

\[
a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos mx \, dx \quad (m = 1, 2, \ldots).
\]

By a similar procedure,

\[
b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin mx \, dx \quad (m = 1, 2, \ldots).
\]

The following box summarizes our discussion and contains basic definitions.

\textbf{EULER FORMULAS FOR THE FOURIER COEFFICIENTS}

Suppose that \(f\) has the Fourier series representation

\[
f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).
\]

Then the coefficients \(a_0, a_n,\) and \(b_n\) are called the Fourier coefficients of \(f\) and are given by the following Euler formulas:

(2) \[
a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx,
\]

(3) \[
a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \quad (n = 1, 2, \ldots),
\]