Theorem 1. *The number of ring homomorphisms from* $Z_{m_1} \times \cdots \times Z_{m_r}$ *into* Z_{p^k} *is given by*

$$1 + N_{p^k}(m_1, \ldots, m_r),$$

where $N_{p^k}(m_1, \ldots, m_r)$ *is the number of elements in the set* $\{m_1, \ldots, m_r\}$ *that are divisible by* p^k.

Proof: Let $\varphi\colon Z_{m_1} \times \cdots \times Z_{m_r} \to Z_{p^k}$ be a ring homomorphism. Then φ is completely determined by $\varphi(e_1), \ldots, \varphi(e_r)$ where e_i is the r-tuple with 1 in the ith component and 0's elsewhere. These are idempotent in Z_{p^k} and hence each must be either 0 or 1. Also, if $\varphi(e_i) = \varphi(e_j) = 1$ for $i \neq j$, then one obtains the contradiction

$$0 = \varphi(0) = \varphi(e_i e_j) = \varphi(e_i)\varphi(e_j) = 1 \cdot 1 = 1.$$

Thus if φ is not the zero homomorphism, then $\varphi(e_i) = 1$ for exactly one value i, and moreover for that i, p^k must divide m_i. Thus, $h(Z_{m_1} \times Z_{m_2} \times \cdots \times Z_{m_r}, Z_{p^k}) = 1 + N_{p^k}(m_1, m_2, \ldots, m_r)$, where $N_{p^k}(m_1, m_2, \ldots, m_r)$ is the number of elements in the set $\{m_1, m_2, \ldots, m_r\}$ that are divisible by p^k.

Theorem 2. *The number of ring homomorphisms from* $Z_{m_1} \times \cdots \times Z_{m_r}$ *into* $Z_{p_1^{k_1}} \times \cdots \times Z_{p_s^{k_s}}$, *where* p_i, $1 \leq i \leq s$, *are primes not necessarily distinct, is*

$$\prod_{i=1}^{s} \left(1 + N_{p^{k_i}}(m_1, m_2, \ldots, m_r)\right).$$

Formulas for the number of ring homomorphisms from rings of the form $Z_m[w]$ *into* $Z_n[w]$, where w is a primitive root of unity, are given in [**2**] and [**3**].

REFERENCES

1. J. A. Gallian and J. Van Buskirk, The number of homomorphisms from Z_m into Z_n, *Amer. Math. Monthly* 91 (1984) 196–197.
2. J. A. Gallian and D. S. Jungreis, Homomorphisms from $Z_m[i]$ into $Z_n[i]$ and $Z_m[\rho]$ into $Z_n[\rho]$, where $i^2 + 1 = 0$ and $\rho^2 + \rho + 1 = 0$, *Amer. Math. Monthly* 95 (1988) 247–249.
3. D. Poulakis, On the homomorphisms of a family of finite rings, *Bull. Greek Math. Soc.* 30 (1989) 43–47.

Birzeit University, P.O.Box 14, Birzeit, West Bank, Palestine.
mohammad@math.birzeit.edu
hasan@math.birzeit.edu

A Simple Proof of a Theorem of Schur

M. Mirzakhani

In 1905, I. Schur [**3**] proved that the maximum number of mutually commuting linearly independent complex matrices of order n is $\lfloor n^2/4 \rfloor + 1$. Forty years later, Jacobson [**2**] gave a simpler derivation of Schur's Theorem and extended it from algebraically closed fields to arbitrary fields. We present a simpler proof of this theorem.

Theorem.. *The maximum number of mutually commuting linearly independent matrices of order n over a field F is* $\lfloor n^2/4 \rfloor + 1$.

Proof: By induction on n, we prove that the maximum number of linearly independent matrices of order n is at most $\lfloor n^2/4 \rfloor + 1$. For $n = 1$, there is nothing to prove. Assume that the theorem is true for $n - 1$ and let $\mathfrak{F}$ be a commuting family of $n \times n$ matrices over F with more than $\lfloor n^2/4 \rfloor + 1$ linearly independent matrices. Without loss of generality one can assume that F is algebraically closed, and so there exists a nonsingular matrix P with entries in F such that $P^{-1}\mathfrak{F}P$ is an upper-triangular family of matrices [**1**, p. 207]. The vector space spanned by the set $P^{-1}\mathfrak{F}P$ consists of commuting upper-triangular matrices. Let us denote it by V and assume that $\dim V \geq \lfloor n^2/4 \rfloor + 2$.

Let $\{A_1, A_2, \ldots, A_{\lfloor n^2/4 \rfloor + 2}\}$ be a linearly independent subset of V. Since for each $i = 1, \ldots, \lfloor n^2/4 \rfloor + 2$, A_i is upper-triangular, there exists an $(n-1) \times (n-1)$ matrix M_i and a $1 \times n$ matrix N_i such that

$$A_i = \left[\begin{array}{c|c} \multicolumn{2}{c}{N_i} \\ \hline 0 & \\ 0 & M_i \\ \vdots & \\ 0 & \end{array}\right].$$

On the other hand, since the A_i's commute, then for $1 \leq i, j \leq \lfloor n^2/4 \rfloor + 2$, we have $M_iM_j = M_jM_i$. Suppose W is the vector space spanned by the set $\{M_1, \ldots, M_{\lfloor n^2/4 \rfloor + 2}\}$ and let $k = \dim W$. Using the induction hypothesis, we have $k \leq \lfloor (n-1)^2/4 \rfloor + 1$. Without loss of generality, assume that W is spanned by the linearly independent matrices $M_1, \ldots, M_k$. Therefore for each i, there exist scalars $n_{i1}, \ldots, n_{ik}$ in F such that $M_i = \sum_{j=1}^{k} n_{ij}M_j$. For $i > k$ we define $B_i = A_i - \sum_{j=1}^{k} n_{ij}A_j$. But the B_i's are linearly independent and for any $i = k+1, \ldots, \lfloor n^2/4 \rfloor + 2$, $B_i = \begin{bmatrix} t_i \\ 0 \end{bmatrix}$, where t_i is a $1 \times n$ matrix. Furthermore the vectors t_i must be linearly independent over F. By a similar argument we obtain a set of linearly independent $n \times 1$ matrices, $t'_{s+1}, \ldots, t'_{\lfloor n^2/4 \rfloor + 2}$, such that $s \leq \lfloor (n-1)^2/4 \rfloor + 1$, and for each i, $B'_i = [0 \mid t'_i]$ is a matrix in V. Now, since all B_i's and B'_i's belong to the commuting family V, one can easily see that $t_it'_j = 0$ for $i = k+1, \ldots, \lfloor n^2/4 \rfloor + 2$ and $j = s+1, \ldots, \lfloor n^2/4 \rfloor + 2$.

Let A be a $(\lfloor n^2/4 \rfloor - k + 2) \times n$ matrix such that for any i, its i-th row is t_i. Since the t_i's are linearly independent, we have $rank A \geq \lfloor n^2/4 \rfloor - k + 2$. On the other hand, $At'_j = 0$ for $j = s+1, \ldots, \lfloor n^2/4 \rfloor + 2$ and the t'_j's are linearly independent. Since $rank A + nullity A = n$, considering the cases that n is odd or even shows that

$$n \geq 2\left(\left\lfloor \frac{n^2}{4} \right\rfloor - \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + 1\right) \geq 2\left\lfloor \frac{n}{2} \right\rfloor + 2$$

which is a contradiction.

The set $\mathfrak{F} = \{E_{ij} \mid 1 \leq i \leq \lfloor n/2 \rfloor, \lfloor n/2 \rfloor + 1 \leq j \leq n\} \cup \{I\}$, where $E_{ij} = [\delta_{pi}\delta_{qj}]$, $1 \leq p, q \leq n$, is a commuting family of $n \times n$ matrices with exactly $\lfloor n^2/4 \rfloor + 1$ linearly independent matrices.

ACKNOWLEDGMENT. The author is grateful to Dr. S. Akbari for his useful comments in the preparation of this note.

REFERENCES

1. K. Hoffman and R. Kunze, *Linear Algebra*, Prentice-Hall, Inc. 1971.
2. N. Jacobson, Schur's Theorems on Commutative Matrices, *Bull. Amer. Math. Soc.* 50 (1944), 431–436.
3. I. Schur, Zur Theorie der Vertauschbären Matrizen, *J. Reine Angew. Math.* 130 (1905) 66–76.

Sharif University of Technology, Tehran, IRAN
m_khani@rose.ipm.ac.ir

From the MONTHLY 75 years ago...

THE DECEMBER MEETING OF THE TEXAS SECTION

The second annual meeting of the Texas Section of the Association was held in the Sunday School room of the First Presbyterian Church in Houston, Texas, on December 1–2, 1922, in conjunction with a meeting of the Texas State Teachers' Association... There were forty-two in attendance including... sixteen members of the Association.

The following papers were read:

(1) "Address on the principles of relativity" by Professor H. HALPERIN;

(2) "Operations with negative numbers, formal and intuitional justification" by Miss LEL RED (by invitation);

(3) "A series of rational functions analogous to Fourier series" by Professor W. P. UDINSKI (by invitation);

(4) "Some applications of Dunhamel's theorem" by Professor H. J. ETTLINGER;

(5) "Descriptive geometry with applications to axomometry and photogrammetry" by Miss ELIZABETH DICE;

(6) "Significant figures" by Professor A. A. BENNETT;

(7) "Interest and annuities" by Professor E. H. JONES;

(8) "History and theory of workmen's compensation insurance" by Mr. C. P. ROCKWELL;

(9) "Simple examples of variable annuities" by Professor L. R. FORD;

(10) "Training in mathematics as preparation for studies in our schools of commerce" by Professor L. H. FLECK (by invitation);

(11) "Mathematical principles of economics" by Professor G. C. EVANS.

Abstracts of these papers follow... .

2. Miss Red explained some of the psychological difficulties involved in the teaching of negative numbers in a first course in algebra. She gave numerous familiar examples that appeal to the scholar which suggest the feasibility of introducing negative numbers and which at the same time indicate the rules of operation that must be adopted if negative numbers are to prove a useful concept.

3. The system of difference equations and boundary conditions analogous to that which in the differential case leads to Fourier series is found to lead to an expansion problem in the difference calculus involving rational functions. The region of convergence is determined in the ordinary sense and under Cesàro summability of given order... .

MONTHLY 30 (1923) 414–415