MATH 2270-3 Fall 2019

First Midterm Exam

Professor: Y.P. Lee

LAST NAME ______________________________

First NAME ______________________________

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INSTRUCTION: Show all of your work. Make sure your answers are clear and legible. Use *specified* method to solve the question. It is not necessary to simplify your final answers.

Problem 1	15	______
Problem 2	15	______
Problem 3	15	______
Problem 4	10	______
Problem 5	15	______
Problem 6	15	______
Problem 7	15	______
Total	100	______

PROBLEM 1 15

Let $S := \{x + iy\sqrt{2} \mid x, y \in \mathbb{R}\}$. Show that S is a subfield of $\mathbb{C}$. (The axioms of fields will be written on the board.)

Axioms of field.

1. $\alpha + \beta = \beta + \alpha \quad \forall \alpha, \beta \in S$
2. $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma \quad \forall \alpha, \beta, \gamma \in S$
3. $\exists!\ \overset{\text{(zero)}}{0} \in S$ s.t. $\alpha + 0 = 0 \quad \forall \alpha \in S$
4. For each $\alpha \in S$, $\exists!\ -\alpha \in S$ s.t. $\alpha + (-\alpha) = 0$.
5. $\alpha\beta = \beta\alpha \quad \forall \alpha, \beta \in S$
6. $\alpha(\beta\gamma) = (\alpha\beta)\gamma \quad \forall \alpha, \beta, \gamma \in S$
7. $\exists!\ \overset{\text{(one)}}{1} \in S$ s.t. $1 \cdot \alpha = \alpha \quad \forall \alpha \in S$
8. For each $\alpha \in S$, $\exists!\ \alpha^{-1} \in S$ s.t. $\alpha \cdot \alpha^{-1} = 1$
9. $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma \quad \forall \alpha, \beta, \gamma \in S$.

Check

- We define the "zero" in S by $0 + i0\sqrt{2} = 0$.
- we define the "one" in S by $1 + i0\sqrt{2} = 1$
- For given $x + iy\sqrt{2} \in S$, we define $-(x + iy\sqrt{2})$ by $-x + i(-y)\sqrt{2}$
- For given $x + iy\sqrt{2} \in S$ we define $(x + iy\sqrt{2})^{-1}$ by $\dfrac{x}{x^2+2y^2} - i\dfrac{y}{x^2+2y^2}\sqrt{2}$

Now check all the 9 axioms by direct computations.

PROBLEM 2 15

Let

$$A := \begin{bmatrix} 3 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 0 \end{bmatrix}.$$

For which triples (y_1, y_2, y_3) does the system $AX = Y$ have a solution?

We compute the row echelon form for A.

$$\begin{bmatrix} 3 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 0 \end{bmatrix} \xrightarrow{\text{row}(2) - \frac{2}{3}\text{row}(1)} \begin{bmatrix} 3 & -1 & 2 \\ 0 & \frac{5}{3} & -\frac{1}{3} \\ 1 & -3 & 0 \end{bmatrix}$$

$$\xrightarrow{\text{row}(3) - \frac{1}{3}\text{row}(1)} \begin{bmatrix} 3 & -1 & 2 \\ 0 & \frac{5}{3} & -\frac{1}{3} \\ 0 & -\frac{8}{3} & -\frac{2}{3} \end{bmatrix} \xrightarrow{\text{row}(3) + \frac{8}{5}\text{row}(2)} \begin{bmatrix} 3 & -1 & 2 \\ 0 & \frac{5}{3} & -\frac{1}{3} \\ 0 & 0 & -\frac{6}{5} \end{bmatrix}$$

We conclude that A is invertible. Hence for any triples (y_1, y_2, y_3), the system $AX = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ has a solution.

PROBLEM 3 15

Prove or disprove that

$$A; = \begin{bmatrix} 1 & 4 & 6 & 8 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 8 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

is invertible. Find A^{-1} if it exists.

A is invertible since all its diagonal entries are nonzero.

$$\left[\begin{array}{cccc|cccc} 1 & 4 & 6 & 8 & 1 & 0 & 0 & 0 \\ 0 & 2 & 3 & 4 & 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 8 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-2\,\text{row}\,2]{\text{row}\,1} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -2 & 0 & 0 \\ & 2 & 3 & 4 & 0 & 1 & 0 & 0 \\ & & 3 & 8 & 0 & 0 & 1 & 0 \\ & & & 4 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[\text{row}\,3 - 2\,\text{row}\,4]{\text{row}\,2 - \text{row}\,3} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -2 & 0 & 0 \\ & 2 & 0 & -4 & & 1 & -1 & 0 \\ & & 3 & 0 & & & 1 & -2 \\ & & & 4 & & & & 1 \end{array}\right] \xrightarrow{\text{row}\,2 + \text{row}\,4} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -2 & 0 & 0 \\ & 2 & 0 & 0 & & 1 & -1 & 1 \\ & & 3 & 0 & & & 1 & -2 \\ & & & 4 & & & & 1 \end{array}\right]$$

$$\xrightarrow{\frac{\text{row}\,2}{2},\ \frac{\text{row}\,3}{3},\ \frac{\text{row}\,4}{4}} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 0 & & & \frac{1}{3} & -\frac{2}{3} \\ 0 & 0 & 0 & 1 & & & & \frac{1}{4} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{3} & -\frac{2}{3} \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}$$

PROBLEM 4 10

Let $V := \{f : \mathbb{R} \to \mathbb{C}\}$ be the set of all complex-valued functions f on the real line such that (for all $t \in \mathbb{R}$) $f(-t) = \overline{f(t)}$. Show that V, together with the operations

$$(f+g)(t) = f(t) + g(t)$$
$$(cf)(t) = c(f(t)),$$

where $c \in \mathbb{C}$, is a not vector space over $\mathbb{C}$.

Take $c = i$, compute

$$\overline{(if)(t)} = \overline{i(f(t))}.$$
$$= -i(\overline{f(t)}).$$
$$= -i(f(-t))$$
$$= -(if)(-t) \quad (\neq (if)(-t))$$

We conclude that.

$f \in V$ but $if \notin V$

Hence it is not a vector field

PROBLEM 5 | 5

Let V be a vector space over a field F. Let W_1 and W_2 be subspaces of V such that $W_1 + W_2 = V$ and $W_1 \cap W_2 = \{0\}$. Prove that for each vector $\alpha \in V$ there are unique vectors $\alpha_1 \in W_1$ and $\alpha_2 \in W_2$ such that $\alpha_1 + \alpha_2 = \alpha$.

Since $W_1 + W_2 = V$, for each vector $\alpha \in V$, there exists $\alpha_1 \in W_1$ and $\alpha_2 \in W_2$ s.t.

$$\alpha_1 + \alpha_2 = \alpha \quad \text{—— (1)}$$

Assume we have $\alpha_1' \in W_1$ and $\alpha_2' \in W_2$ s.t.

$$\alpha_1' + \alpha_2' = \alpha \quad \text{—— (2)}$$

Considre (1) − (2)

$$\alpha_1 - \alpha_1' = -(\alpha_2 - \alpha_2') \in W_1 \cap W_2 = \{0\}$$

Hence $\alpha_1 = \alpha_1'$ and $\alpha_2 = \alpha_2'$. This proves the uniqueness

PROBLEM 6 15

Let $F \subset \mathbb{C}$ be a subfield. Suppose u, v, w are linearly independent vectors in V. Prove that $u - v$, $v - w$, w ~~$-u$~~ are linearly independent.

We have $c_1 u + c_2 v + c_3 w = 0 \Rightarrow c_1 = c_2 = c_3 = 0$.

Now consider

$$a_1(u-v) + a_2(v-w) + a_3 w = 0$$

$$\Rightarrow a_1 u + (a_2 - a_1) v + (a_3 - a_2) w = 0.$$

$$\Rightarrow \begin{cases} a_1 = 0 \\ a_2 - a_1 = 0 \\ a_3 - a_2 = 0 \end{cases} \Rightarrow \begin{cases} a_1 = 0 \\ a_2 = 0 \\ a_3 = 0 \end{cases}$$

We conclude that all relations for $u - v$, $v - w$, w are trivial.

Hence they are linearly independent.

PROBLEM 7 15

Find a basis for the subspace of $\mathbb{R}^4$ spanned by the following 4 vectors
$v_1 = (1,1,2,4)$, $v_2 = (2,-1,-5,2)$, $v_3 = (1,-1,-4,0)$, $v_4 = (2,1,1,6)$.

We compute the row echelon form for

$$\begin{bmatrix} 1 & 1 & 2 & 4 \\ 2 & -1 & -5 & 2 \\ 1 & -1 & -4 & 0 \\ 2 & 1 & 1 & 6 \end{bmatrix} \xrightarrow[\text{row}(4) - 2\cdot\text{row}(1)]{\text{row}(2) - 2\cdot\text{row}(1),\ \text{row}(3) - \text{row}(1)} \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & -3 & -9 & -6 \\ 0 & -2 & -6 & -4 \\ 0 & -1 & -3 & -2 \end{bmatrix}$$

$$\xrightarrow[\text{row}(4) - \frac{1}{3}\text{row}(2)]{\text{row}(3) - \frac{2}{3}\text{row}(2)} \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & -3 & -9 & -6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow$$

$(1,1,2,4)$, $(0,-3,-9,-6)$ can form a basis.