EULER CHARACTERISTICS OF UNIVERSAL COTANGENT LINE BUNDLES ON $\overline{M}_{1,n}$

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ABSTRACT. An effective algorithm of computing $\chi(\overline{M}_{1,n}, L_1^\otimes d_1 \otimes \cdots \otimes L_n^\otimes d_n)$ is given. Here $\overline{M}_{1,n}$ is the moduli stack of $n$-pointed stable curves of genus one and $L_i$ are the universal cotangent line bundles.

In addition, a simple proof of a genus zero vanishing theorem, first proved by R. Pandharipande, is presented.

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0. INTRODUCTION

Let $\overline{M}_{1,n}$ be the moduli stack of $n$-pointed genus 1 stable curves. Let $\mathcal{H}$ be the Hodge bundle on $\overline{M}_{1,n}$, and $L_i$, $1 \leq i \leq n$ the universal cotangent line bundles at the $i$-th marked point. The main result of this paper is the following theorem.

**Theorem 0.1.** There is an effective algorithm of computing the Euler characteristics

$$\chi_{d,d_1,\ldots,d_n} := \chi(\overline{M}_{1,n}, \mathcal{H}^\otimes -d \otimes \prod_{i=1}^n L_i^\otimes d_i), \quad d, d_i \geq 0.$$
This work continues the early result [7] at genus zero. One of our motivations of this work is to provide some basic ingredients for quantum K-theory [8]: In the calculation of quantum K-invariants at genus one via localization, \( \chi_{d, \delta_1, \ldots, \delta_n} \) will naturally appear.

\( \chi_{d, \delta_1, \ldots, \delta_n} \) can also be viewed as the natural K-theretic counterpart of the Witten–Kontsevich correlators at genus one, where the correlators are of the form

\[
\int_{\mathcal{M}_{1, n}} \psi_1^{d_1} \cdots \psi_n^{d_n}.
\]

We note that the integration is the pushforward to a point (i.e. the spectrum of the ground field) in the cohomology theory, while the Euler characteristic is the corresponding operation in the K-theory. Adding the Hodge bundle in \( \chi_{d, \delta_1, \ldots, \delta_n} \) corresponds to the Hodge integrals.

Our strategy of proving Theorem 0.1 is, very roughly, to apply the orbifold Riemann-Roch theorem to

\[
\chi' := \chi \left( \mathcal{M}_{1, n}, \mathcal{O} \otimes -d \prod_{i=1}^n (L_i^{\otimes d_i} - \mathcal{O}) \right), \quad d, d_i \geq 0.
\]

By carefully examining and performing computations on the twisted sectors, we were able to determine the \( \chi' \). In doing so, we find the use of generating functions essential. See Section 2, starting with Equation (2.1). The majority of the hard calculation lies here. It is then not difficult to see that one can determine \( \chi \) on \( \mathcal{M}_{1, n} \) by \( \chi' \) and \( \chi \) on \( \mathcal{M}_{1, n-1} \). Hence, we can reduce all calculations to \( n = 1 \) case, which we compute explicitly.

Indeed, the \( n = 1 \) case is closely related to the theory of modular forms. Theorem 0.1 can be considered as a vast generalization of the following well-known fact below.

**Proposition 0.2** (See Lemma 2.7 and Proposition 2.8).

\[
\chi \left( \mathcal{M}_{1, 1}, \mathcal{O} \otimes \frac{1}{1 - q^1} \right) = \frac{1}{(1 - q^4)(1 - q^6)}.
\]

Since \( \mathcal{M}_{1, 1} \) is the moduli stack of elliptic curves, and sections of \( L_i^{\otimes k} \) are the modular forms of weight \( 2k \). Proposition 0.2 can be considered as a rephrase of the classical result that the space of the modular forms are generated by a weight four and a weight six modular forms.

Another result included in this paper, in Appendix B, is a new proof of Pandharipande’s vanishing theorem [10] at genus zero.

**Theorem 0.3** ([10]).

\[
H^i(\overline{\mathcal{M}}_{0, n}, \otimes_{i=1}^n L_i^{d_i}) = 0
\]

for \( i \geq 1 \) and \( d_i \geq 0 \).
Our proof is comparably much simpler and shorter, and do not use M. Kapranov’s results on $\overline{\mathcal{M}}_{0,n}$. Only basic definitions and elementary manipulation of spectral sequences are used.

This note is organized as follows. In Section 1 we recall the necessary background on the structure of the inertia stack of $\overline{\mathcal{M}}_{1,n}$, mostly quoting [9], with supplemental details added in Appendix A. We then formulate a more precise version of the reduction algorithm in Section 2, and prove Theorem 0.1 there. In Appendix B the (new) proof of Theorem 0.3 is given.

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1. **Preliminaries**

We work over the ground field $\mathbb{C}$.

1.1. **Twisted sectors of $\overline{\mathcal{M}}_{1,n}$**. We summarize the results concerning the inertia stack of $\overline{\mathcal{M}}_{1,n}$. Essentially, all these results are proved in [9]. As the arguments in [9] about the inertia stack of $\overline{\mathcal{M}}_{1,n}$ are sometimes sketchy, we try to provide more details in this subsection and in appendix A.

**Notations.** Let’s start with the general notations. For a stack $\mathcal{X}$, we denote by $I\mathcal{X}$ its inertia stack, defined by the (2-)Cartesian diagram

$$
\begin{array}{ccc}
I\mathcal{X} & \longrightarrow & \mathcal{X} \\
\downarrow & & \downarrow \\
\mathcal{X} & \xrightarrow{\Delta} & \mathcal{X} \times \mathcal{X}.
\end{array}
$$

In other words, $I\mathcal{X}$ consists of pairs $(x, g)$ where $x \in \mathcal{X}$ and $g \in Aut_{\mathcal{X}}(x)$. We use $I(\mathcal{X}, r)$ to denote the component of $I\mathcal{X}$ consisting of pairs $(x, g)$ with $g \in Aut_{\mathcal{X}}(x)$ of order $r$. By a component, we mean an open and closed substack.

$I\mathcal{X}$ is the disjoint union of its connected component. There is a distinguished component of $I\mathcal{X}$, which is isomorphic to $\mathcal{X}$ and is denoted by $(\mathcal{X}, \text{Id})$. The rest of the connected components are the twisted sectors.

The open immersion $\mathcal{M}_{1,n} \hookrightarrow \overline{\mathcal{M}}_{1,n}$ induces an open immersion $I\mathcal{M}_{1,n} \hookrightarrow I\overline{\mathcal{M}}_{1,n}$ (cf. Proposition A.4), so a twisted sector of $I\mathcal{M}_{1,n}$ can be viewed as an open substack of $I\overline{\mathcal{M}}_{1,n}$. Take the closure of this open substack (cf. Definition A.5), we get a connected closed substack of $I\overline{\mathcal{M}}_{1,n}$.

Let $K \subset \{1, 2, \cdots, n\}$, with $|K| \geq 2$, and $K^c$ its complement. The closed immersion

$$
\Delta_K : \overline{\mathcal{M}}_{0,K \cup \bullet} \times \overline{\mathcal{M}}_{1,K^c \cup \bullet} \longrightarrow \overline{\mathcal{M}}_{1,n}
$$

gluing the marked points corresponding to $\bullet$ induces a closed immersion:

$$
I\Delta_K : I((\overline{\mathcal{M}}_{0,K \cup \bullet} \times \overline{\mathcal{M}}_{1,K^c \cup \bullet})) \longrightarrow I\overline{\mathcal{M}}_{1,n}
$$

(cf. Proposition A.4). A twisted sector of $I((\overline{\mathcal{M}}_{0,K \cup \bullet} \times \overline{\mathcal{M}}_{1,K^c \cup \bullet})$ can be viewed as a closed substack of $I((\overline{\mathcal{M}}_{1,n})$ via $I\Delta_K$. 
These two types of closed substacks give us all the twisted sectors of $\overline{\mathcal{M}}_{1,n}$. We summarize the above discussions in the following theorem.

**Theorem 1.1** ([9]). The twisted sectors of $\overline{\mathcal{M}}_{1,n}$ come from either of the following two sources:

1. the closure of a twisted sector of $\mathcal{M}_{1,n}$ in $\overline{\mathcal{M}}_{1,n}$, or
2. A twisted sector of $\mathcal{I}(\mathcal{M}_{0,K∪\bullet} × \mathcal{M}_{1,K∪\bullet})$ via $IΔ_K$.

**Remark 1.2.** This theorem follows from the knowledge of Spec $C$ points of the stacks involved. Generally, given a finite collection of connected closed substacks $\{X_i, i \in I\}$ of a DM stack $X$ which is reduced and of finite type over $C$, the induced map from the disjoint union $\bigsqcup_{i \in I} X_i$ to $X$ is an isomorphism if it is universally bijective.

By the theorem above, $\overline{\mathcal{M}}_{1,n}$ is determined inductively from the twisted sectors of $\mathcal{M}_{1,k}$, $k ≤ n$.

When $n ≥ 5$, $\mathcal{M}_{1,n}$ has no twisted sectors, so we need to determine the twisted sectors of $\mathcal{M}_{1,n}$ and their closure in $\overline{\mathcal{M}}_{1,n}$ when $n ≤ 4$.

We start with an description of $\overline{\mathcal{M}}_{1,1}$, which is in the literature in various forms.

**Proposition 1.3.** $\overline{\mathcal{M}}_{1,1}$ is equivalent to the weighted projective space $\mathbb{P}(4,6)$.

**Remark 1.4.** This equivalence can be realized as follows ([9]). Let $U = \mathbb{A}^2 - (0,0)$, with $C^*$ action: $\lambda \cdot (a,b) = (\lambda^4a, \lambda^6b)$, where $\lambda \in C$, $(a,b) \in U$. The equivalence from $\mathbb{P}(4,6) := [U/C^*]$ to $\overline{\mathcal{M}}_{1,1}$ is induced from an $C^*$-equivariant family of 1 pointed genus one stable curves $C → U$, where

$$C = \{(a,b) × [x : y : z] ∈ U × \mathbb{P}^2 | y^2z = x^3 + axz^2 + bz^3\},$$

with section

$$s : U → U × [0, 1, 0] ⊂ C,$$

the $C^*$ action is given by

$$\lambda \cdot ((a,b) × [x : y : z]) = (\lambda^4a, \lambda^6b) × [\lambda^2x : \lambda^3y : z].$$

From this description of $\overline{\mathcal{M}}_{1,1}$, we see that for a 1-marked smooth elliptic curve the order of an automorphism is 2, 3, 4, or 6.

- An order 2 automorphism is an elliptic involution.
- Order 4 automorphism comes from the curve

$$C_4 : \{[x : y : z] ∈ \mathbb{P}^2 | y^2z = x^3 + xz^2\} \text{ with } [0 : 1 : 0] \text{ marked.}$$

The order 4 automorphism is given by

$$\lambda \cdot [x : y : z] = [\lambda^2x : \lambda^3y : z] \text{ for } \lambda = i \text{ or } -i.$$
Theorem 1.7

Remark

For \( (1) \) given \( (2) \)

Given \( x \in \mathcal{X}(\text{Spec C}) \) and an order \( r \) element \( g \in \text{Aut}_X(x) \), the pair \( (x, g) \) determines a representable morphism from \( B_{\mu_r} \) to \( \mathcal{X} \). (cf. [1] 3.2) The restriction of the natural map \( q_\mathcal{X} : I\mathcal{X} \to \mathcal{X} \) forgetting the automorphism to a dimension 0 twisted sector of the form \( B_{\mu_r} \) can thus be determined by a pair \( (x, g) \).

As \( I\mathcal{M}_{1,n} = \sqcup_r I(\mathcal{M}_{1,n}, r) \), the theorem determines \( I(\mathcal{M}_{1,n}, r) \) for \( r > 2 \). When \( r > 2 \), it can be proved that \( I(\mathcal{M}_{1,n}, r) \) is zero dimensional, and a connected, smooth, zero dimensional DM stack, of finite type over \( C \), is of the form \( BG \) for some finite group \( G \).

For \( \mathcal{A}_r \), let \( \overline{\mathcal{A}_r} \) be its closure in \( I\mathcal{M}_{1,n} \). We have the following description of the structures of \( \overline{\mathcal{A}_r} \).

**Theorem 1.7 ([9])**

1. \( \overline{\mathcal{A}_1} \) is isomorphic to \( \overline{\mathcal{M}_{1,1}} \).
• $\overline{A}_2 \subset \overline{\mathcal{M}}_{1,2}$ is isomorphic to $\mathbb{P}(2,4)$.
• $\overline{A}_3 \subset \overline{\mathcal{M}}_{1,3}$ is isomorphic to $\mathbb{P}(2,2)$.
• $\overline{A}_4 \subset \overline{\mathcal{M}}_{1,4}$ is isomorphic to $\mathbb{P}(2,2)$.

(2) $\overline{A}_i, 2 \leq i \leq 4$ does not intersect with the boundary divisors $\Delta_K$.

Remark 1.8. In [9], the isomorphism between $\overline{A}_k$ and a weighted projective space are proved using Remark 1.4, Proposition A.1 and A.2.

Remark 1.9. Since $B\mu_r$ is proper and $\overline{\mathcal{M}}_{1,n}$ is separated, it is easy to see, for $n < 5$, the zero dimensional twisted sectors of $\mathcal{M}_{1,n}$ are closed in $\overline{\mathcal{M}}_{1,n}$.

1.2. Riemann-Roch formula for Stacks. We recall the Riemann-Roch formula in a version needed for this paper.

Theorem 1.10 ([4, 11]). Let $X$ be a smooth, proper Deligne-Mumford stack with quasi-projective coarse moduli space, $E$ a vector bundle on $X$. Assume $X$ has the resolution property, i.e. every coherent sheaf is a quotient of a vector bundle, then we have the following formula for the Euler characteristics of $E$:

$$\chi(X, E) = \int_{I^X} \overline{Ch}(E) \overline{Td}(q_\lambda^{-1} N_{I^X/X})$$

Here
• $I^X$ is the inertial stack of $X$, with projection $q_\lambda : I^X \to X$.
• $\overline{Ch}(E) \in H^*(I^X)$ is the Chern character of the bundle $\rho(q_\lambda^* E)$.
• $\rho(F) := \sum \xi F(\xi) \in K^0(I^X)$, $F(\xi)$ is the eigenbundle of $F$ with eigenvalue $\xi$.
• $\overline{Td}(q_\lambda^{-1} N_{I^X/X}) = \frac{Td(I^X)}{Ch(\rho_\lambda^{-1} N_{I^X/X})}$, where $Td$ and $Ch$ are the usual Todd class and Chern character. $N_{I^X/X}$ is the normal bundle for $q_\lambda$, and $N^\vee$ is the dual of $N$.
• $\lambda_1(V) := \sum_{a \geq 0} (-1)^a \Lambda^a V$ is the $\lambda_1$ operation in K-theory. If $V = \bigoplus V_i$ is direct sum of line bundles $V_i$, then $\lambda_1(V) = \prod_i (1 - V_i)$.

Remark 1.11. $\overline{\mathcal{M}}_{1,n}$ satisfies the resolution property. See, e.g. [6].

1.3. String Equation.

Proposition 1.12. Let $\pi : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n-1}$ be the forgetful map forgetting the nth marked point, $g \geq 1$ then

$$\pi_*(\mathcal{H}^{\otimes -d} \prod_{i<n} L_i^{\otimes d_i}) = \mathcal{H}^{\otimes -d} \prod_{i<n} L_i^{\otimes d_i} \otimes (O - \mathcal{H}^{-1} + \sum_{i<n} \sum_{0 \leq m \leq d_i} L_i^{\otimes m}).$$

Here $\mathcal{H}$ is the Hodge bundle; $L_i$ are the universal cotangent line bundles; $d, d_i \geq 0$ are nonnegative integers; $\pi_*$ is the K-theoretic pushforward.

Proof. For the universal family $\pi : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n-1}$ on $\overline{\mathcal{M}}_{g,n-1}$, we have sections

$$s_i : \overline{\mathcal{M}}_{g,n-1} \to \overline{\mathcal{M}}_{g,n}.$$
This is a closed immersion. Let $D_i$ be the boundary divisor of $\overline{M}_{g,n}$ determined by $s_i$.

Since $L_i = \pi^*L_i + D_i$, $\mathcal{H} = \pi^*\mathcal{H}$,
\[
\pi_*(\mathcal{H}^{-d} \prod_{i \leq n} L_i^{d_i}) = \mathcal{H}^{-d} \prod_{i \leq n} L_i^{d_i} \otimes \pi_*(\mathcal{O}(\sum_{i \leq n} d_i D_i)).
\]

On $\overline{M}_{g,n}$, start with the exact sequence
\[
0 \to \mathcal{O}(-D_i) \to \mathcal{O} \to \mathcal{O}_{D_i} \to 0.
\]
Tensoring with any divisor $E$, we have
\[
0 \to \mathcal{O}(E - D_i) \to \mathcal{O}(E) \to \mathcal{O}_{D_i}(E) \to 0.
\]
Therefore,
\[
\pi_*(\mathcal{O}(E)) = \pi_*(\mathcal{O}(E - D_i)) + \pi_*(\mathcal{O}_{D_i}(E)).
\]

Note that $\mathcal{O}_{D_i}(D_i)$ is the normal bundle of $D_i$ in $\overline{M}_{g,n}$. If we identify $D_i$ with $\overline{M}_{g,n-1}$ via $s_i$, then for $d \geq 0$,
\[
\pi_*(\mathcal{O}_{D_i}(dD_i)) = \pi_*s_js_i^*\mathcal{O}_{D_i}(dD_i) = s_i^*\mathcal{O}_{D_i}(D_i)^{\otimes d} = L_i^d.
\]

Now for $E = \sum_{i \leq n} a_i D_i$. If $a_j > 0$ we have
\[
\pi_*(\mathcal{O}(E)) = \pi_*(\mathcal{O}(E - D_j)) + \pi_*(\mathcal{O}_{D_j}(E)) = \pi_*(\mathcal{O}(E - D_j)) + L_j^{\otimes a_j}.
\]

From here, it is easy to see
\[
\pi_*(\mathcal{O}(\sum_{i \leq n} d_i D_i)) = \pi_*\mathcal{O} + \sum_{i \leq n} \sum_{0 \leq m \leq d_i} L_i^{\otimes m} = \mathcal{O} - \mathcal{H}^{-1} + \sum_{i \leq n} \sum_{0 \leq m \leq d_i} L_i^{\otimes m}
\]
as $\pi_*\mathcal{O} = \pi_*\mathcal{O} - R^1\pi_*\mathcal{O} = \mathcal{O} - (\pi_*\omega)^\vee = \mathcal{O} - \mathcal{H}^{-1}$. Therefore,
\[
\pi_*(\mathcal{H}^{-d} \prod_{i \leq n} L_i^{d_i}) = \mathcal{H}^{-d} \prod_{i \leq n} L_i^{d_i} \otimes (\mathcal{O} - \mathcal{H}^{-1} + \sum_{i \leq n} \sum_{0 \leq m \leq d_i} L_i^{\otimes m}).
\]

2. Euler characteristics of universal cotangent line bundles

Let $\mathcal{O}$ be the structure sheaf, $\mathcal{H}$ the Hodge bundle, and $L_i$ the cotangent line bundle corresponding to the $i$th marked point on $\overline{M}_{1,n}$. In this section, we explain the calculation of

\[
\chi\left(\overline{M}_{1,n}, \mathcal{H}^{-d} \prod_{i=1}^n L_i^{d_i}\right), d, d_i \geq 0.
\]

It is more efficient to encode these numbers into a generating function

\[
X_n := \chi\left(\overline{M}_{1,n}, \frac{1}{1-q\mathcal{H}^{-1}} \prod_{i=1}^n \frac{1}{1-q_i L_i}\right)
\]
\[
= \sum_{d, d_i \geq 0} q^d \prod_{i=1}^n q_i^{d_i} \chi\left(\overline{M}_{1,n}, \mathcal{H}^{-d} \prod_{i=1}^n L_i^{d_i}\right).
\]
(2.1)
We will first show that the calculation of $X_n$ can be reduced to that of $X_{n-1}$ if another generating function $\Phi_n$ in (2.2) can be calculated. We explicitly determine $\Phi_n$ and $X_1$ via Riemann–Roch (Theorem 1.10).

2.1. Reduction from $\overline{M}_{1,n}$ to $\overline{M}_{1,n-1}$. Let $\Phi_n$ be their generating function

$$\Phi_n := \chi\left(\overline{M}_{1,n}, \mathcal{H}^{\otimes d} \prod_{i=1}^{n} (L_i^{\otimes d_i} - \mathcal{O})\right), \quad d, d_i \geq 0.$$  

\textbf{Lemma 2.1.} When $n > 1$, $X_n$ is determined by $\Phi_n$ and $X_{n-1}$.

\textbf{Proof.} Expand $\prod_{i=1}^{n} (L_i^{\otimes d_i} - \mathcal{O})$, we see that

$$
\chi\left(\overline{M}_{1,n}, \mathcal{H}^{\otimes d} \prod_{i=1}^{n} (L_i^{\otimes d_i})\right)
= \chi\left(\overline{M}_{1,n}, \mathcal{H}^{\otimes d} \prod_{i=1}^{n} (L_i^{\otimes d_i} - \mathcal{O})\right)
- \sum_{I \subset \{n\}, \emptyset \neq I} (-1)^{|I|} \chi\left(\overline{M}_{1,n}, \mathcal{H}^{\otimes d} \prod_{i \in I} L_i^{\otimes d_i}\right).
$$

We now show that the RHS of this equation is determined by $\Phi_n$ and $X_{n-1}$. The first term on the RHS is determined by $\Phi_n$. For each term $\chi(\overline{M}_{1,n}, \mathcal{H}^{\otimes d} \prod_{i \notin I} L_i^{\otimes d_i})$, we apply the string equation (Theorem 1.12):

$$
\chi(\overline{M}_{1,n}, \mathcal{H}^{\otimes d} \prod_{i \notin I} L_i^{\otimes d_i})
= \chi(\overline{M}_{1,n-1}, \pi_*(\mathcal{H}^{\otimes d} \prod_{i \notin I} L_i^{\otimes d_i}))
= \chi(\overline{M}_{1,n-1}, \mathcal{H}^{\otimes d} \prod_{i \notin I} L_i^{\otimes d_i} \otimes (\mathcal{O} - \mathcal{H}^{-1} + \sum_{0 \leq m \leq d_i} L_i^{\otimes m})).
$$

The last line is obviously determined by the coefficients of $X_{n-1}$. In the second line $\pi_*$ denotes the $K$-theoretic pushforward of the proper map $\pi$, where $\pi : \overline{M}_{1,n} \rightarrow \overline{M}_{1,n-1}$ is any forgetful map forgetting a marked point in $I$. This proves the lemma.

It follows immediately that we have the corollary below.

\textbf{Corollary 2.2.} For $n > 1$, $X_n$ is determined by $\{\Phi_k, 1 < k \leq n\}$ and $X_1$.  

2.2. Calculation of $\Phi_n$. By the Riemann-Roch formula (Theorem 1.10), we have

$$
\Phi_n = \int_{\overline{M}_{1,n}} \widetilde{Ch}\left(\frac{1}{1-qH^{-1}} \prod_{i=1}^{n} \left(\frac{1}{1-q_iL_i} - \frac{1}{1-q_iO}\right)\right) \widetilde{Td}(q^*T\overline{M}_{1,n})
$$

As the inertia stack $I\overline{M}_{1,n}$ is the disjoint union of the distinguished component $(\overline{M}_{1,n}, \text{Id})$ and its twisted sectors. The integral is the sum of the contributions from these components.

**Proposition 2.3.** The contribution to $\Phi_n$ from $(\overline{M}_{1,n}, \text{Id})$ is

$$
\frac{(n-1)!}{24(1-q)} \prod_{i=1}^{n} \frac{q_i}{(1-q_i)^2}.
$$

**Proof.** On $(\overline{M}_{1,n}, \text{Id})$, $\widetilde{Ch}, \widetilde{Td}$ reduces to the usual $\text{Ch}, \text{Td}$, and

$$
\text{Ch}\left(\frac{1}{1-qH^{-1}}\right) = \frac{1}{1-q}(1 + \text{degree} > 0 \text{ terms}),
$$

$$
\text{Ch}\left(\frac{1}{1-q_iL_i} - \frac{1}{1-q_iO}\right) = \frac{q_i}{(1-q_i)^2} c_1(L_i)(1 + \text{degree} > 0 \text{ terms}).
$$

Applying the dilaton equation

$$
\int_{\overline{M}_{1,n}} c_1(L_1) \cdots c_1(L_n) = n \int_{\overline{M}_{1,n-1}} c_1(L_1) \cdots c_1(L_{n-1}),
$$

and

$$
\int_{\overline{M}_{1,1}} c_1(L_1) = \frac{1}{24},
$$

We can now evaluate the integral

$$
\int_{\overline{M}_{1,n}} \text{Ch}\left(\frac{1}{1-qH^{-1}} \prod_{i=1}^{n} \left(\frac{1}{1-q_iL_i} - \frac{1}{1-q_iO}\right)\right) \text{Td}(\overline{M}_{1,n})
$$

$$
= \frac{(n-1)!}{24(1-q)} \prod_{i=1}^{n} \frac{q_i}{(1-q_i)^2}.
$$

The proposition is proved. $\square$

**Proposition 2.4.** The contribution to $\Phi_n$ from the twisted sectors of type (2) in Theorem 1.1, i.e. from $I\Delta_K$, is zero.

**Proof.** By our construction, such a twisted sector is the product of $\overline{M}_{0,KU^*}$ and a twisted sector $\mathcal{I}$ of $\overline{M}_{1,KU^*}$.

The natural map $\overline{M}_{0,KU^*} \times \mathcal{I} \to \overline{M}_{1,n}$ factors through

$$
\Delta_K : \overline{M}_{0,KU^*} \times \overline{M}_{1,KU^*} \to \overline{M}_{1,n}.
$$

By using the known results:

- The dual of the normal bundle for $\Delta_K$ is $pr_i^* (L^*) \otimes pr_i^* (L^*)$. Here $pr_i$ is the projection of $\overline{M}_{0,KU^*} \times \overline{M}_{1,KU^*}$ onto its $i$th factor.
• \( \Delta_K(L_i) = pr_1^*(L_i) \) for \( i \in K \), and is \( pr_2^*(L_i) \) for \( i \notin K \).

• \( \Delta_K(H) = pr_2^*(H) \).

It is easy to see the integrand
\[
\tilde{Ch}(\frac{1}{1-qH} \prod_{i=1}^n (\frac{1}{1-q_i L_i} - \frac{1}{1-q_i O})) \tilde{Td}(q^* \overline{M}_{1,n})
\]
over \( \overline{M}_{0,K,\bullet} \times I \) is a linear combination of classes that is the cross product of a class on \( I \) and a class on \( \overline{M}_{0,K,\bullet} \).

Such a class on \( \overline{M}_{0,K,\bullet} \) has a factor of \( \prod_{i \in K} c_1(L_i) \), which comes from
\[
\tilde{Ch}(\prod_{i \in K} (\frac{1}{1-q_i L_i} - \frac{1}{1-q_i O})).
\]
As the degree of \( \prod_{i \in K} c_1(L_i) \) exceeds the dimension of \( \overline{M}_{0,K,\bullet} \), the integral is zero. 

\( \square \)

**Proposition 2.5.** For \( 2 \leq k \leq 4 \), the contribution from \( \overline{A}_k \) is

\[
(-1)^k \frac{1}{24(1+q)} \prod_{i=1}^k \frac{q_i}{1-q_i^2} \cdot (11 + \frac{2q}{1+q} - \sum_{i=1}^n \frac{2q_i}{1+q_i}) \cdot d_k
\]

here \( d_k = 6, 6, 3 \) for \( k = 4, 3, 2 \), respectively. It is the degree of the map \( \overline{A}_k \rightarrow \overline{M}_{1,1} \)
forgetting all but one marked point.

**Proof.** Over \( \overline{A}_k \), the eigenvalues of the bundles appeared in the integrand must be \(-1\), as the nontrivial automorphism is of order 2.

We have
\[
\tilde{Ch}(\frac{1}{1-qH} \prod_{i=1}^k (\frac{1}{1-q_i L_i} - \frac{1}{1-q_i O}))
\]
\[
= \frac{1}{1+q e^{c_1(H^{-1})}} \prod_i (\frac{1}{1+q_i e^{c_1(L_i)}} - \frac{1}{1-q_i})
\]
\[
= \frac{1}{1+q - q c_1(H)} \prod_i (\frac{1}{1+q_i + q_i c_1(L_i)} - \frac{1}{1-q_i})
\]
\[
= \frac{1}{1+q} (1 + \frac{q}{1+q} c_1(H)) \prod_i (\frac{1}{1+q_i} (1 - \frac{q_i}{1+q_i} c_1(L_i)) - \frac{1}{1-q_i})
\]
\[
= (-2)^k \frac{1}{1+q} \prod_i \frac{q_i}{1-q_i^2} \left( 1 + \frac{q}{1+q} c_1(H) + \sum_i \frac{1-q_i}{2(1+q_i)} c_1(L_i) \right),
\]

and
\[
\text{Ch} \left( \rho \circ (\Lambda_{-1}(N^\vee_{\overline{A}_k/\overline{M}_{1,k}})) \right) = 2^{k-1} \left( 1 + \frac{1}{2} c_1(N^\vee_{\overline{A}_k/\overline{M}_{1,k}}) \right).
\]
There is no higher degree terms as $\mathcal{A}_k$ is 1 dimensional. Thus
\[
\int_{\mathcal{A}_k} \tilde{\chi}(1 - qH - \frac{1}{1 - qL_i}) \partial_1 d(q^* (T\mathcal{M}_{1,1} |_{\mathcal{A}_k})) \\
= (-1)^k \frac{1}{1 + q} \prod \frac{q_i}{1 - q_i^2} \cdot \left( 2q_i \int_{\mathcal{A}_k} c_1(H) + \sum_i \frac{1 - q_i}{1 + q_i} \int_{\mathcal{A}_k} (L_i) + \int_{\mathcal{A}_k} c_1(T\mathcal{M}_{1,1} |_{\mathcal{A}_k}) \right).
\]
Here we have used
\[-c_1(N_{\mathcal{A}_k/\mathcal{M}_{1,1}}) + 2T \partial_1 (\mathcal{A}_k) = c_1(T\mathcal{M}_{1,1} |_{\mathcal{A}_k}).\]
Hence we are left to determine
\[
\int_{\mathcal{A}_k} c_1(L_j |_{\mathcal{A}_k}), j \leq k, \text{ and } \int_{\mathcal{A}_k} c_1(T\mathcal{M}_{1,1} |_{\mathcal{A}_k}).
\]
To find these two integrals, we consider the map $\mathcal{A}_k \to \mathcal{M}_{1,1}$ forgetting all but one marked point, and express the integrand as the pullback of some class on $\mathcal{M}_{1,1}$.

Now we move on to $\mathcal{M}_{1,n|\bullet}$. Let $\pi : \mathcal{M}_{1,n|\bullet} \to \mathcal{M}_{1,n}$ be the forgetful morphism, forgetting the marked point $\bullet$. We have
\[
L_j = \pi^* L_j + \Delta_{\{j\}}, 1 \leq j \leq n
\]
and
\[
L_\bullet = \omega_\pi + \sum_{1 \leq j \leq n} \Delta_{\{j\}},
\]
where $\omega_\pi$ is the relative dualizing sheaf for $\pi$, which in this case is
\[
\omega_\pi = \omega_{\mathcal{M}_{1,n|\bullet}} \otimes \pi^* \omega_{\mathcal{M}_{1,n}}^{-1}.
\]
Take the first Chern class $c_1$ of these equations we get
\[
c_1(L_j) = \pi^* c_1(L_j) + \Delta_{\{j\}},
\]
\[
c_1(T\mathcal{M}_{1,n|\bullet}) = \pi^* c_1(T\mathcal{M}_{1,n}) - c_1(L_\bullet) + \sum_{1 \leq j \leq n} \Delta_{\{j\}}.
\]
Now apply this to the forgetful map
\[
\mathcal{A}_k \subset \mathcal{M}_{1,k} \to \mathcal{M}_{1,1}.
\]
Recall that $\mathcal{A}_k$ doesn’t intersect with the boundary divisors of the form $\Delta_{\{j\}}$, we have
\[
\int_{\mathcal{A}_k} c_1(L_j) = d_k \int_{\mathcal{M}_{1,1}} c_1(L_1), 1 \leq j \leq k,
\]
\[
\int_{\mathcal{A}_k} c_1(T\mathcal{M}_{1,k} |_{\mathcal{A}_k}) = (11 - k) d_k \int_{\mathcal{M}_{1,1}} c_1(L_1).
\]
Here we used the facts:

- Under $\mathcal{M}_{1,1} \simeq \mathbb{P}(4,6)$, $L_1 \simeq \mathcal{O}(1)$. 

\[ T\mathbb{P}(4, 6) \simeq \mathcal{O}(1)^{10}. \]
\[ \mathcal{H} \simeq \pi^* \mathcal{H}. \]
\[ \mathcal{H} \simeq L_1 \text{ on } \overline{\mathcal{M}}_{1,1}. \]

These facts are all easy to see from the basic definitions, except possibly the second, which can be shown via the Euler sequence on the weighted projective stack \( \mathbb{P}(4, 6) \)
\[ \mathcal{O} \to \mathcal{O}(4) \oplus \mathcal{O}(6) \to T\mathbb{P}(4, 6) \to 0. \]

To get \( d_k \), we pass to the induced map between the coarse moduli spaces, and count the number of points in a general fiber of this induced map. \( \square \)

**Proposition 2.6.** The contribution to \( \Phi_n \) from the integral over zero dimensional twisted sectors are the following.

- **the contribution from** \((C_4, i) \sqcup (C_4, -i)\) is
  \[ \frac{1}{4} \prod_{j=1,2} q_j \cdot \frac{1 - q + q_1 + q_2 - q_1 q_2 + q q_1 + q q_2 + q q_1 q_2}{(1 + q^2)(1 + q_1^2)(1 + q_2^2)}. \]

- **the contribution from** \((C_6^4, \omega_6^2) \sqcup (C_6^4, / \omega_6^4)\) is
  \[ \frac{1}{3} \prod_{j=1,2} q_j \cdot \frac{1 - q + (q + 2)(q_1 + q_2) + (2q + 1)q_1 q_2}{(1 + q + q^2)(1 + q_1 + q_1^2)(1 + q_2 + q_2^2)}. \]

- **the contribution from** \((C_6^4, \omega_6^3) \sqcup (C_6^4, / \omega_6^4)\) is
  \[ \frac{1}{3} \prod_{j=1,2,3} q_j \cdot \frac{1 - q + (q + 2)(q_1 + q_2 + q_3) + (2q + 1)(q_1 q_2 + q_1 q_3 + q_2 q_3) + (q - 1)q_1 q_2 q_3}{(1 + q + q^2)(1 + q_1 + q_1^2)(1 + q_2 + q_2^2)(1 + q_3 + q_3^2)}. \]

**Proof.** To simplify the notation, we will use \((C, \lambda)\) to denote a twisted sector. On a zero dimensional twisted sector, only the eigenvalues of the bundles involved in the integrand is relevant.

The fiber of \( L_j \) at a smooth \( n \) pointed curve \( \{C, p_1, p_2, \ldots, p_n\} \) is \( T^* p^*_j C \), so the eigenvalue is determined by the action of the automorphism on this cotangent space. It is \( \lambda \) on \((C, \lambda)\). The eigenvalue for \( \mathcal{H} \) is the same as above, as it is the pullback of \( L_1 \) on \( \overline{\mathcal{M}}_{1,1} \) via the forgetful map.

For \((C_4, \lambda), (C_6, \lambda)\) of \( \overline{\mathcal{M}}_{1,1} \), explicit calculation tells us the eigenvalue of \( T^* \overline{\mathcal{M}}_{1,1} \) is \( \lambda^2 \). It is then not hard to show, using the forgetful map to \( \overline{\mathcal{M}}_{1,1} \), that the eigenvalues of \( T^* \overline{\mathcal{M}}_{1,2} \) are \( \lambda, \lambda^2 \) for \((C_4, \lambda)\) and \((C_6, \lambda)\), the eigenvalues of \( T^* \overline{\mathcal{M}}_{1,3} \) are \( \lambda, \lambda, \lambda^2 \) for \((C_6, \lambda)\).

On the twisted sector \((C, \lambda)\) of \( \overline{\mathcal{M}}_{1,n} \),
\[ \widetilde{Ch}( \frac{1}{1 - qH - L_i} \prod_{i \leq n} \frac{1}{1 - q_iL_i} - \frac{1}{1 - q_iO}) \]

\[ = \frac{1}{1 - q\lambda - \prod_{i \leq n} (1 - q_i)} \]

\[ = \frac{1}{1 - q\lambda - \prod_{i \leq n} (1 - q_i)} \]

\[ = (\lambda - 1)^n \frac{1}{1 - q\lambda - \prod_{i \leq n} (1 - q_i)} \]

and

\[ \widetilde{Td}(T\mathcal{M}_{1,n}|_{(C, \lambda)}) = \frac{1}{(1 - \lambda^2)(1 - \lambda)^{n-1}} = \frac{1}{(1 + \lambda)(1 - \lambda)^n}. \]

The sum of the integral on \((C_4', i) \sqcup (C_4, -i)\) is then

\[ \frac{1}{4} \sum_{\lambda = i, -i} \frac{1}{(1 - q\lambda - \prod_{i \leq n} (1 - q_i)} \frac{1}{1 - q_i} \frac{1}{1 - q_i\lambda}, \]

which equals

\[ \frac{1}{4} \prod_{j=1,2} \frac{q_j}{1 - q_j} \cdot \frac{1 - q + q_1 + q_2 - q_1q_2 + qq_1 + qq_2 + qqq_1q_2}{(1 + q^2)(1 + q^2_1)(1 + q^2_2)}. \]

We have the factor \(\frac{1}{4}\) as we are integrating twisted sectors isomorphic to \(B\mu_4\).

The remaining cases also follow directly from our formula of \(\widetilde{Ch}, \widetilde{Td}\). \(\Box\)

2.3. Calculation for \(X_1\). Under the isomorphism \(\mathcal{M}_{1,1} \simeq \mathbb{P}(4, 6), H, L_1\) all corresponds to \(O(1)\), so

\[ \chi(\mathcal{M}_{1,1}, H \otimes -d \otimes L_1^{\otimes 1} = \chi(\mathbb{P}(4, 6), O(d_1 - d)), \]

and we see \(X_1\) is determined by \(\chi(\mathbb{P}(4, 6), O(k)), k \in \mathbb{Z}\).

Lemma 2.7. Let \(h^0(O(k)) = \text{dim}_C H^0(\mathbb{P}(4, 6), O(k))\), then

\[ \sum_{k=0}^{\infty} h^0(O(k))q^k = \frac{1}{(1 - q^4)(1 - q^6)^{'}}, \]

and \(h^0(O(k)) = 0 \text{ if } k < 0.\)

Proof. The section of \(O(k)\) on \(\mathbb{P}(4, 6)\) correspondences to polynomials \(f(x, y) \in C[x, y]\) such that, \(f(\lambda^4x, \lambda^6y) = \lambda^k f(x, y)\) for any \(\lambda \in C^*\). From this description, it is easy to see \(\text{dim}_C H^0(\mathbb{P}(4, 6), O(k))\) is the number of monomials


$$x^a y^b$$ such that $4a + 6b = k$, or put in another way, the coefficient of $q^k$ in the power series $rac{1}{(1 - q^4)(1 - q^6)}$.

**Proposition 2.8.**

$$\chi(\mathcal{M}_{1,n}) = \frac{1}{1-q}\frac{1}{1-q_1 L_1} = \frac{(1+q)(1-q^4-q^6-q_1 q^8-q_1^2q^8-q_1^3q^8-q_1^4 q^8+q^4q_1^2+q^6q_1^2+q^8q_1^2)}{(1-q^4)(1-q^6)(1-q_1^4)(1-q_1^6)}$$

**Proof.** This follows from the lemma above. By Serre duality

$$H^1(\mathbb{P}(4,6), \mathcal{O}(k)) \simeq H^0(\mathbb{P}(4,6), \mathcal{O}(-10-k))^\vee,$$

so $\chi(\mathbb{P}(4,6), \mathcal{O}(k)) = h^0(\mathcal{O}(k)) - h^0(\mathcal{O}(-k-10)).$

\[\square\]

**APPENDIX A. INERTIA STACK OF $\mathcal{M}_{1,n}$**

We show that $\mathcal{M}_{1,n}$ is a quotient stack, and describe the inertia stack of a quotient stack. These known results are central to the determination of $\mathcal{I}\mathcal{M}_{1,n}$, and we supply the proofs here for lacking of a convenient reference. We then explain the constructions related to Theorem 1.1.

If an algebraic group $G$ acts on a scheme $U$, the group action will be on the right, and we will use $\sigma$ to denote such an action, $[U/G]$ to denote the quotient stack.

**Proposition A.1.** Let $U_{n+1} \rightarrow U_n$ be a $C^*$-equivariant family of $n$-pointed curves. If the induced map from $[U_n/C^*]$ to $\mathcal{M}_{1,n}$ is an isomorphism, then $\mathcal{M}_{1,n+1}$ is isomorphic to the quotient stack $[U_{n+1}/C^*]$.

**Proof.** An equivariant family is the same as a cartesian diagram

$$\begin{array}{ccc}
U_n \times C^* & \xrightarrow{\sigma} & U_n \\
\downarrow^{pr_1} & & \downarrow \\
U_n & \rightarrow & \mathcal{M}_{1,n}
\end{array}$$

here the map $U_n \rightarrow \mathcal{M}_{1,n}$ corresponds to the family $U_{n+1} \rightarrow U_n$. This diagram induces a map between groupoids $U_n \times C^* \Rightarrow U_n$, and $U_n \times_{\mathcal{M}_{1,n}} U_n \Rightarrow U_n$, hence a morphism $[U_n/C^*]$ to $\mathcal{M}_{1,n}$. More explicitly, given $B \rightarrow [U_n/C^*]$ or a $C^*$ bundle $E$ over $B$ with an equivariant map $f : E \rightarrow U_n$, pulling back the family $U_{n+1}$ on $U_n$ to $E$ via $f$, $G$ equivariance ensues that this family on $E$ descends to a family over $B$ and this is the map $[U_n/C^*](B) \rightarrow \mathcal{M}_{1,n}(B)$.

$\mathcal{M}_{1,n+1}$ being the universal curve, a map $B \rightarrow \mathcal{M}_{1,n+1}$ is determined by a family of $n$-pointed curve $C$ over $B$, plus an extra section for this family, and the map $U_{n+1} \rightarrow \mathcal{M}_{1,n+1}$ is given by the family which is obtained by pulling back the family $U_{n+1}$ over $U_n$ to $U_{n+1}$, plus the diagonal section.
Now a family of \( n \)-pointed curve \( C \) over \( B \) comes from an equivariant family over a \( C^\ast \) bundle \( E \) over \( B \), recall the family on \( E \) is the pull back family via \( f \). The extra section for \( C \) would be an extra equivariant section of \( E \), this amounts to an equivariant map \( E \to [U_{n+1}/C^\ast] \), and from here it is not hard to show this gives an equivalence between \( [U_{n+1}/C^\ast] \) and \( \overline{\mathcal{M}}_{1,n+1} \).

Proposition A.2. For a quotient stack \( [U/G] \), its inertial stack \( I([U/G]) \) is \( [U'/G] \). Here \( U' \subset U \times G \) consists of \( (u, g) \) such that \( u \cdot g = u \), or more precisely, \( U' \) fits in a cartesian diagram

\[
\begin{array}{ccc}
U' & \longrightarrow & U \times G \\
\downarrow & & \downarrow (e, pr_1) \\
U & \underset{\Delta U}{\longrightarrow} & U \times U
\end{array}
\]

and the \( G \) action on \( U' \) is the one induced from its action on \( U \times G \) by \( (u, g) \cdot h = (uh, h^{-1}gh) \). The projection \( I([U/G]) \to [U/G] \) is induced from the \( G \) equivariant map \( U' \to U \times G \) \( pr_1 \to U \).

Proof. We sketch the proof. A morphism \( B \to I([U/G]) \) corresponds to a principal \( G \) bundle \( \pi : E \to B \) with a \( G \) equivariant map \( f : E \to U \) and an automorphism \( \alpha \) of \( E \), such that \( f \circ \alpha = f \).

Consider pulling back \( E \) via \( \pi \), we have a diagram

\[
\begin{array}{ccc}
E \times G & \longrightarrow & E \\
\downarrow & & \downarrow \\
E & \underset{\pi}{\longrightarrow} & B
\end{array}
\]

\( \alpha \) pulls back to an automorphism \( \beta \) of \( E \times G \) over \( E \) that descends to \( E \). Now an automorphism of the trivial \( G \) bundle over \( E \) is given by a map \( \theta : E \to G \), i.e. \( \beta(e, g) = (e, \theta(e)g) \), so we get a map \( (f, \theta) : E \to U \times G \) which actually determines an element of \( [U'/G](B) \). In fact, \( f \circ \alpha = f \) is equivalent to \( g \circ \beta = g \), and this holds iff \( (f, \theta) \) factors through \( U' \). \( \beta \) descends to \( a \) iff \( (f, \theta) \) is \( G \) equivariant.

Remark A.3. Note that this proposition tells us a twisted sector of \( I\overline{\mathcal{M}}_{1,n} \) is always a closed substack of \( \overline{\mathcal{M}}_{1,n} \).

A map between stacks \( f : \mathcal{X} \to \mathcal{Y} \) induces naturally a map \( If : I\mathcal{X} \to I\mathcal{Y} \). we have a commutative diagram

\[
\begin{array}{ccc}
I\mathcal{X} & \underset{If}{\longrightarrow} & I\mathcal{Y} \\
\downarrow & & \downarrow \\
\mathcal{X} & \underset{f}{\longrightarrow} & \mathcal{Y}
\end{array}
\]
If diagram (†) is cartesian, then for any \( S \to X \) determined by \( x \in \mathcal{X}(S) \), it is easy to see \( \text{Aut}_X(x) \) is isomorphic to \( \text{Aut}_Y(f(x)) \).

Conversely, we have

**Proposition A.4.** Given a representable morphism \( f: \mathcal{X} \to \mathcal{Y} \), where \( \mathcal{X}, \mathcal{Y} \) are DM stacks of finite type over \( \mathbb{C} \). Assume \( f \) is unramified, and the induced map from \( \text{Aut}_X(x) \) to \( \text{Aut}_Y(f(x)) \) is an isomorphism for all \( x \in \mathcal{X}(\text{Spec}\mathbb{C}) \), then (†) is cartesian. In particular, if \( f \) is an open or closed immersion, so is \( \text{I}f \).

**Proof.** The proof comes from the following cartesian diagrams

\[
\begin{array}{ccc}
I\mathcal{X} & \longrightarrow & \mathcal{X} \
\downarrow & & \downarrow \\
\mathcal{X} \times Y I\mathcal{Y} & \longrightarrow & \mathcal{X} \times Y \\
\downarrow & & \downarrow \\
\mathcal{X} & \longrightarrow & \mathcal{Y} \\
\end{array}
\]

Since \( f \) is representable and unramified, the vertical map \( \mathcal{X} \to \mathcal{X} \times Y \) is an open immersion. So \( I\mathcal{X} \to \mathcal{X} \times Y \) is an open immersion.

To conclude it is actually an isomorphism, we need to show it is universally bijective. This can be checked at \( \text{Spec}\mathbb{C} \) points, and it is equivalent to the assumption that the map from \( \text{Aut}_X(x) \) to \( \text{Aut}_Y(f(x)) \) is an isomorphism for all \( x \in \mathcal{X}(\text{Spec}\mathbb{C}) \).

\( \square \)

**Definition A.5.** For an open substack \( Z \) of an algebraic stack \( \mathcal{X} \), the closure \( \overline{Z} \) of \( Z \) in \( \mathcal{X} \), is the closed substack of \( \mathcal{X} \) such that, for any smooth surjective map \( X \to \mathcal{X} \) with \( X \) a scheme,

we have a cartesian diagram

\[
\begin{array}{ccc}
Z & \longrightarrow & \overline{Z}_{\text{red}} \\
\downarrow & & \downarrow \\
Z & \longrightarrow & \overline{Z} \\
\end{array}
\]

where \( \overline{Z}_{\text{red}} \) is the reduced subscheme of \( X \) whose underlying space is the closure of \( Z \) in \( X \).

**Remark A.6.** \( \overline{Z} \) is determined by a groupoid presentation of \( \mathcal{X} \).

**Appendix B. A simple proof of Pandharipande’s vanishing theorem**

The purpose of this appendix is to give a very simple and self-contained proof of Theorem 0.3, first proved in [10]. Recall that the theorem states that at genus zero

\[ H^i(\overline{\mathcal{M}}_{0,n}, \otimes_{i=1}^n L_i^d) = 0 \]

(B.1)
for \( i \geq 1 \) and \( d_i \geq 0 \).

1

We will prove (B.1) by induction on \( n \). Note that \( \overline{\mathcal{M}}_{0,3} \) is a point, (B.1) obviously holds.

When \( n > 3 \), we have two cases:

1. \( d_i \geq 1 \) for all \( i \).
2. one of the \( d_i \) is zero.

For the first case, let \( V := \bigoplus_{i=1}^{n} L_i \). Choose generic sections \( s_i \) of \( L_i \) such that the zero loci of \( s_i \) are in general position. Consider the Koszul complex

\[
0 \rightarrow \mathcal{O} \xrightarrow{d} V \xrightarrow{d} \bigwedge^2 V \rightarrow \cdots \xrightarrow{d} \bigwedge^n V \rightarrow 0,
\]

where the differential \( d \) is defined as \( d := \sum_{i=1}^{n} s_i \wedge \) with \( s_i \) considered as a section of \( V \). Because of our choice of \( s_i \), this complex is exact. Tensor this complex with \( \bigotimes_{i=1}^{n} L_i^{d_i-1} \), we get a resolution of \( \bigotimes_{i=1}^{n} L_i^{d_i-1} \) as \( \bigwedge^n V \otimes \bigotimes_{i=1}^{n} L_i^{d_i-1} \) is just \( \bigotimes_{i=1}^{n} L_i^{d_i-1} \), and \( H^*(\overline{\mathcal{M}}_{0,n}, \bigotimes L_i^{d_i}) \) can be computed using hypercohomology.

Consider the double complex \( (C^p(\mathcal{U}, \mathcal{K}^q); \delta, d)_{p,q \geq 0, q<n} \), where \( \mathcal{U} \) is a covering of \( \overline{\mathcal{M}}_{0,n}, \mathcal{K}^q := \bigwedge^q V \otimes \bigotimes_{i=1}^{n} L_i^{d_i-1} \), \( C^p \) are the Čech cochain groups, \( \delta \) is the Čech differential.

Using two canonical filtrations (by \( p \) and \( q \) respectively), we obtain two spectral sequences \( ^{'}E_r^{p,q} \) and \( ^{''}E_r^{p,q} \) with

\[
^{'}E_1^{p,q} = H^p(\overline{\mathcal{M}}_{0,n}, \mathcal{K}^q),
\]

\[
^{''}E_2^{p,q} = H^p(\overline{\mathcal{M}}_{0,n}, H_q^*(\mathcal{K}^*))
\]

and these two spectral sequences abut to the same hypercohomology \( H^*(\overline{\mathcal{M}}_{0,n}, \mathcal{K}^*) \).

---

1The method presented in this appendix can also be used to compute \( H^0(\overline{\mathcal{M}}_{0,n}, \bigotimes L_i^{d_i}) \). It is also hoped that this method can help to produce an \( S_n \)-equivariant version of our genus zero formula [7], which is needed in the quantum \( K \)-theory [8] computation of general target spaces.
By induction, \( E^{p,q}_1 = 0 \) if \( p \neq 0 \), since \( K^q \) is the direct sum of \( \otimes L^{d_i} \)'s with \( \sum d'_i < \sum d_i \). So \( E^{p,q}_1 \) degenerates at \( r = 1 \), and \( H^q(\overline{M}_{0,n}, K^*) = 0 \) when \( q \geq n \), as \( E^{0,q}_1 = 0, q \geq n \) by our construction.

Note that \( "E^{p,q}_2 \) is zero for \( q \neq n - 1 \), and \( "E^{p,n-1}_2 = H^p(\overline{M}_{0,n} \otimes L_i^{d_i}) \), \( "E^{p,q}_r \) degenerates at \( r = 2 \). Therefore, \( H^p(\overline{M}_{0,n} \otimes L_i^{d_i}) = H^{p+n-1}(\overline{M}_{0,n}, K^*) = 0 \), when \( p + n - 1 \geq n \), or \( p \geq 1 \).

For the case that some \( d_i = 0 \), say \( d_n = 0 \),

\[
H^i(\overline{M}_{0,n} \otimes \sum_{i=1}^{n-1} L_i^{d_i}) = H^i(\overline{M}_{0,n-1}, R^0 \pi_*(\otimes_{i=1}^{n-1} L_i^{d_i}))
= H^i(\overline{M}_{0,n-1}, (\otimes_{i=1}^{n-1} L_i^{d_i}) \otimes R^0 \pi_*(\mathcal{O}(\sum_{i<n} d_i D_i)))
= H^i(\overline{M}_{0,n-1}, (\otimes_{i=1}^{n-1} L_i^{d_i}) \otimes (1 + \sum_{i,d_i \neq 0} \sum_{j=1}^{d_i} L_i^{-j}))
\]

where \( \pi : \overline{M}_{0,n} \to \overline{M}_{0,n-1} \) is the forgetful morphism as before. Note that we have used the same symbols \( L_i \) for line bundles on \( \overline{M}_{0,n} \) and on \( \overline{M}_{0,n-1} \). The first equality used the Leray spectral sequence, as the higher direct image \( R^1 \pi_*(\otimes_{i=1}^{n-1} L_i^{d_i}) = 0 \), which follows from the fact that \( H^1(C, \mathcal{O}_C(D)) = 0 \) for \( C \) rational and degree of \( \mathcal{O}_C(D) \) positive. The third equality used the genus zero string equation, \( R^0 \pi_* \) can be computed in a way similar to the proof of Proposition 1.12.

\[
R^0 \pi_*(\mathcal{O}(\sum_{i<n} d_i D_i)) = 1 + \sum_{i,d_i \neq 0} \sum_{j=1}^{d_i} L_i^{-j}.
\]

It is now clear that \( H^i(\overline{M}_{0,n} \otimes \sum_{i=1}^{n-1} L_i^{d_i}) \) can be written as a summation of \( H^i(\overline{M}_{0,n-1} \otimes L_i^{d_i}) \), which by induction is zero.

This ends our proof.

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