QUANTUM COHOMOLOGY UNDER BIRATIONAL MAPS AND TRANSITIONS

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Abstract. This is the third author’s lecture in String-Math 2015 at Sanya, which reports on our recent works in quantum cohomology. After reviewing the quantum Lefschetz and quantum Leray–Hirsch, we discuss applications to special smooth flops, flips and blow-ups in higher dimensions. For conifold transitions of Calabi–Yau 3-folds, formulations for small resolutions (blow-ups along Weil divisors) are sketched.

0. Introduction

0.1. Classical aspects on algebraic geometry. In the study of algebraic geometry, we usually encounter projective morphisms \( f : Y \to X \), e.g. blow-ups, bundle morphisms etc., and it is a basic question to study relations of geometric quantities under such morphisms.

By its very definition of being projective, there are factorizations of \( f \) into compositions of simpler morphisms in the following form. There are vector bundles \( \mathcal{E} \to X \) and associated projective bundles \( \pi : P = P_X(\mathcal{E}) \to X \) such that \( f = \pi \circ \iota \):

\[
\begin{array}{ccc}
Y & \xrightarrow{\iota} & P \\
\downarrow f & & \downarrow \pi \\
X & & \\
\end{array}
\]

where \( \iota : Y \hookrightarrow P \) is an imbedding. The choice of \( P \) is by no means unique. In fact \( \mathcal{E} \) can be taken to be a trivial bundle, say of rank \( r \), and then \( P = \mathbb{P}^{r-1} \times X \) is a product. However, a good choice of \( P \) is usually important so that the induced imbedding \( \iota \) has good structures.

If \( i(Y) \subset P \) is a complete intersection, namely that there is a split vector bundle \( V = \bigoplus L_i \to P \) and a section \( \sigma \in \Gamma(P, V) \) such that \( i(Y) = \sigma^{-1}(0) \) is its zero loci, then one develops

Lefschetz Hyperplane Theorem

to study relations between \( Y \) and \( P \). In most cases one does not obtain complete intersection imbedding automatically. Nevertheless sometimes one may employ the technique of deformations to the normal cone to reduce the problem under study to such a situation. The most famous one is the
proof of Grothendieck–Riemann–Roch theorem [5]. Historically it is the proof of GRR which lays the foundation of these techniques. Among those many later applications, it is also notable to mention the proof of the invariance of complex elliptic genera under K-equivalent birational maps [23].

For the (projective) bundle map \( \pi \), one develops

\[
\text{Leray–Hirsch Theorem}
\]

to study relations between \( P \) and \( X \). The techniques in this package include direct images, spectral sequences etc., and usually can be applied to more general bundle maps such that the fiber manifold is well understood. A combination of results for \( \iota \) and \( \pi \) then gives the desired result for \( f \).

0.2. Quantum aspects. In this article, following the above flow-chart, we survey the related developments on the quantum cohomology ring \( \mathbb{Q}H(X) \), or equivalently the genus zero Gromov–Witten theory. More precisely we consider the Dubrovin (flat) connection \( \nabla \) on \( TH(X) \) and analyze its behavior under various maps including complete intersection imbedding and projective bundle maps. The essential mathematical tools are the corresponding Quantum Lefschetz Hyperplane Theorem and the Quantum Leray–Hirsch Theorem.

The first version of quantum Lefschetz was proved around 1996 by Lian, Liu, and Yau [19] and by Givental [6] independently (cf. [3]). It is also known as the mirror theorem since its major motivation and application is to prove the counting formula of rational curves on quintic Calabi–Yau 3-folds predicted by Candelas et al. In that setup \( P \) could be a semi-Fano toric manifold and \( V = \bigoplus L_i \) is a sum of convex line bundles such that \( c_1(Y) \geq 0 \) (semi-Fano or sub Calabi–Yau). There are several improvement of quantum Lefschetz afterwards (e.g. [8]). The most general version which allows the background manifold \( P \) to be general and without the condition on \( c_1(Y) \) was obtained by Coates and Givental [2]. We will review this latest theory in terms of Dubrovin connections and introduced the notion of Birkhoff factorizations and generalized mirror transforms.

For quantum Leray–Hirsch, a version of \( P = P_X(\mathcal{E}) \) with \( \mathcal{E} = \mathcal{O} \oplus L \) a rank two split bundle first appeared in the work of Maulik and Pandharipande [20]. A version on the more general case of toric bundle \( P \to X \) build on a split vector bundle \( \mathcal{E} \to X \) was proved by Brown [1]. It had been formulated in the framework of Dubrovin connections and used to prove the invariance of quantum cohomology rings under ordinary flops by Lee, Lin and Wang in 2011 [11, 12]. More recently, together with F. Qu, we proved a quantum splitting principle and remove the splitting assumption in the quantum Leray–Hirsch [9]. For the applications to be discussed here, we will only focus on the Dubrovin connection in the split case. The ideas of naive quantization basis and admissible lift of Mori cone are the essential ingredients to formulate the quantum Leray-Hirsch.
By combining quantum Lefachetz and quantum Leray–Hirsch, we discuss several applications on birational maps including (i) the analytic continuations, i.e. invariance, of $QH$ under ordinary flops, (ii) smooth blow-ups along complete intersection centers, as well as (iii) decomposition theorem of projective local models of simple ordinary flips. The latter two applications are new, and both address the issue on the lack of functoriality of $QH$ under non $K$-equivalent transformations.

Indeed for simple $(r,r')$ flips $f : X \rightarrow X'$, there is an orthogonal decomposition $H(X) = \mathcal{T}^{-1}H(X') \oplus K$ where $\mathcal{T} = [\Gamma_f] : H(X) \rightarrow H(X')$ is the map induced from the graph correspondence with kernel $K \cong C^{r-r'}$ and $\mathcal{T}^{-1} := [\Gamma_f^t]$ is the transpose correspondence. Under a suitable choice of quantum frame $\tilde{T}_i$'s which deforms the classical cohomology basis $T_i$'s, we have a ring isomorphism (decomposition)

$$QH(X) \cong \langle \tilde{T}_1, \ldots, \tilde{T}_{\dim H(X')} \rangle \times C^{r-r'}$$

such that $\langle \tilde{T}_1, \ldots, \tilde{T}_{\dim H(X')} \rangle \cong QH(X')$ as $\mathcal{T}^z$ modules, but not as rings.

In this survey we illustrate this for the case of $(2,1)$ flips.

Comparing with the current developments on string-math related topics, the subjects discussed here are more classical in flavor as there is essentially no higher genus theory nor modern mirror symmetry involved. Mirror symmetry phenomenon happens in the large complex/Kähler structure limit while birational maps are essentially located at finite distance Kähler degenerations. Also we treat only smooth varieties. However, in higher dimensional birational geometry, namely the minimal model theory, it is indispensable to include singularities in the variety under consideration.

It is clear that most of the subjects discussed here can be extended to orbifolds since orbifold Gromov–Witten theory is now well developed. However MMP requires more general singularities than orbifold ones and it is still a long way towards a useful quantum minimal model program. Of course the smooth case is the first step on it, and it is our hope to make progresses on the general case in the near future.

0.3. Towards a QMMP. As the next step, we are led to consider birational maps up to complex deformations, i.e. transitions. There are many technical issues from classic algebraic geometry on this regard. Nevertheless it has become clearer in recent years that it is indispensable to allow certain transitions in the classification of higher dimensional varieties.

For Calabi–Yau 3-folds the famous Ried's fantasy [21] on connecting CY with different topology through transitions is still one of the major research problems in this area. We give a very brief sketch on our recent work [13] in understanding the transition of quantum $\mathcal{A}$ model and $\mathcal{B}$ model in projective conifold transitions $X \rightarrow Y$ through a conifold degeneration $X \rightarrow \Delta$ from $X = X$ to $X = X_0$, followed by a small resolution $Y \rightarrow X$. In particular the notion of linked GW invariants with respect to a set of vanishing spheres $S_i \subset X$ is introduced which corresponds to the non-exceptional
GW invariants on Y. For GW invariants supported on exceptional curves, a basic exact sequence shows that there is a local transition between them and the Yukawa couplings of the vanishing periods.

This article ends at issue on effective computations on the quantum transitions in terms of blow-up formula of GW invariants when the blow-up is along certain non-complete intersection (Weil) divisors.

1. Quantum Cohomology

In this article, unless stated otherwise, all varieties X, Y, P, . . . under considerations are assumed to be smooth and projective over C. The cone of effective one cycles (Mori cone) in X is denoted by \( \text{NE}(X) \).

1.1. Dubrovin connection. A general reference is [3]. We fix a cohomology basis \( T_i \in H = H(X) \), with dual basis \( \{ T^i \} \). A general element in cohomology is denoted by \( t = \sum t^i T_i \). The genus zero GW theory is encoded by its generating function (pre-potential) \( F(t) = \langle \langle \rangle \rangle \), where for \( a_1, \ldots, a_m \in H(X) \),

\[ \langle \langle a_1, \ldots, a_m \rangle \rangle == \sum_{\beta \in \text{NE}(X)} \sum_{n=0}^{\infty} \frac{q_\beta^n}{n!} \langle a_1, \ldots, a_m, t, \ldots, t \rangle_{g=0,m+n,\beta}. \]

The (formal) Novikov variables \( q_\beta \)'s are inserted to avoid the issue on convergence. It also keeps track on the natural weighted structure. In general \( t \) is also treated as a formal variable.

Denote by \( F_{ijk} = \partial_{ijk}^3 F = \langle \langle T_i, T_j, T_k \rangle \rangle \) the 3-point generating functions, and set \( A_{ijk}^k := \sum_l F_{ijl} g^{kl} \). Then the big quantum product at \( t \) is defined by

\[ T_i \ast_i T_j = \sum_k A_{ijk}^k(t) T_k. \]

The product is associative due to the WDVV equations. Equivalently it corresponds to the flatness of the Dubrovin connection on \( TH \otimes \mathbb{C}[q^\bullet] \):

\[ \nabla := d - \frac{1}{z} A \equiv d - \frac{1}{z} \sum_i dt_i \otimes A_i. \]

The special role played by the \( z \) parameter shows that \( \nabla \) is flat if and only if \( dA = 0 = A \wedge A \), which is equivalent to WDVV.

1.2. J function and cyclic \( \mathcal{D}^\mathbb{Z} \) modules. Write \( t = t_0 + t_1 + t_2 \) with \( t_0 \in H^0 \) and \( t_1 \in H^2 \). The generating function of all genus zero GW invariants with at most one descendent insertion is organized as

\[ J(t,z^{-1}) := 1 + \frac{t}{z} + \sum_{\beta,n,i} \frac{q_\beta^n}{n!} T_i \left\langle \frac{T^i}{z(z-\psi)} \right\rangle_\beta (t)^n \]

\[ = e^\frac{t}{z} + \sum_{\beta \neq 0,n,i} \frac{q_\beta^n}{n!} e^{\frac{t_{a+1}}{z} + (1,\beta)} T_i \left\langle \frac{T^i}{z(z-\psi)} \right\rangle_\beta (t_2)^n \]
where the fundamental class axiom (string equation) and the divisor axiom are applied to get the second equality. The important role played by the $J$ function comes from the following quantum differential equation (QDE)

\[(1.2) \quad z\partial_i z\partial_j J = \sum_k A_{ij}^k z\partial_k J,\]

which follows from the topological recursion relation (TRR). It implies that the quantum cohomology $QH(X)$ can be regarded as the cyclic $D$ module $Dz$ with base (frame) given by

\[z\partial_i J \equiv e^{t/z} T_i \pmod{q^*} = T_i + \cdots.\]

It is clear that for $T_{i_0} = 1$ being the fundamental class, $z\partial_{i_0} J = J$.

The ring of differential operators $Dz = \mathbb{C}[t, q^*\{z, z\partial\}]$ is defined so that $p = \sum_\beta q^\beta p_\beta \in Dz$ implies that $p_\beta$ is polynomial in $z$ and $z\partial$.

In practice it is sometimes easier to study the GW theory or the $J$ function on the small parameter space $H^0 \oplus H^2$. The expression of $J$ with $t^2 = 0$ is known as the small $J$ function. Using the divisorial reconstruction theorem in [16], the small $J$ function determines the sub-algebra of $QH(X)$ generated by $H^2(X)$.

2. Review on Quantum Lefschetz

2.1. Quantum Lefschetz for toric base and concavex bundle with $c_1 \geq 0$.

Given a projective manifold $P$ and convex line bundles $L_i \to P$, $1 \leq i \leq r$, and a section $\sigma \in \Gamma(P, \bigoplus_{i=1}^r L_i)$ such that $Y = \sigma^{-1}(0) \hookrightarrow P$ is a smooth submanifold. Given $QH(P)$, the problem is to compute $QH(Y)$.

The method of localizations on stable map moduli spaces leads to the so called factorial trick or hypergeometric modifications. To state it, we start with the cohomology valued factorial

\[(L)_\beta := \prod_{m=1}^{L_\beta} (L + mz)\]

whenever the intersection number $L_\beta \geq 0$. Then we set

\[I^Y(t, z, z^{-1}) := \sum_{\beta \in NE(P)} q^\beta J_\beta^P(t, z^{-1}) \times \prod_{i=1}^r (L_i)_\beta\]

as an approximation of $J^Y$.

When the ambient space $P$ is a semi-Fano toric manifold (i.e. $c_1(P) \geq 0$), Lian–Liu–Yau and Givental used $\mathbb{C}^\times$ localizations to determine $J^P$. (This in turn determines $QH(P)$ since $H(P)$ is generated by divisors.) Over such a semi-Fano toric base, they further proved the

**Theorem 2.1** (Mirror Theorem). [19, 6] For $c_1(Y) \geq 0$, $t \in H^0 \oplus H^2$, we have

\[(I^Y / I^0_Y)(t, z^{-1}) = J^Y(\tau, z^{-1})\]

up to the mirror map $t \mapsto \tau(t)$ which matches $1/z$ coefficients on both sides.
Here $I_0^Y$ is the component of $z^0$ terms. Notice that the assumption $c_1(Y) \geq 0$ implies that $I^Y$ is still an expression in $z^{-1}$.

In [19], the line bundles $L_i$’s are also allowed to be concave. In that case their mirror principle determines a certain type of twisted GW invariants.

### 2.2. Quantum Lefschetz over general base and split bundles.

Notice that without the condition that $c_1(Y) \geq 0$, the approximation $I^Y$ might contain terms with positive $z$ powers. But $J^Y$, by definition, contains only terms in powers of $z^{-1}$. Hence a more sophisticated transformation is needed in order to relate $I^Y$ to $J^Y$.

Coates and Givental considered the following situation: Let $P$ be a general projective manifold whose big quantum cohomology ring $QH(P)$ is given. Let $L_i \rightarrow P$, $1 \leq i \leq r$, be line bundles. They defined twisted GW invariants in this setup. When $L_i$’s are base-point free the twisted invariants are the GW invariants of the complete intersection sub-manifold $Y = \sigma^{-1}(0)$ for a generic section $\sigma \in \Gamma(P, \bigoplus_{i=1}^r L_i)$. They proved

**Theorem 2.2.** [2] Given $I^P(t)$ in $H(P)$, then $I^Y \in \mathcal{D}^P I^P$. More precisely, there exists a linear differential operator $b = b(\tau, z, q^*, z \partial_\bullet)$ which is polynomial in $z$ on each finite truncation of the Novikov variables $q^*$ such that

$$I^Y(t, z, z^{-1}) = b(\tau, z, q^*, z \partial_\bullet) J^Y(\tau, z^{-1}).$$

Here $t \mapsto \tau(t) \in H(X)$ is a transformation determined by this property.

To get a better understanding of the statement, we notice that $b$ can be taken to be linear in $z \partial_\bullet$’s because of the QDE (1.2) on $I^Y$. In a similar fashion, the differentiation $z \partial_i I^Y$ can also be represented by $\sum_j B_{ij} z \partial_j J^Y$ for some formal functions $B_{ij}$’s which are polynomials in $z$ in any $\beta \in NE(P)$. This leads to the so called Birkhoff factorization

$$I^Y(t, z, z^{-1}) = (z \partial_\bullet)(\tau, z^{-1})B(\tau, z)$$

of the square matrix $(z \partial_\bullet I) = (z \partial_1 I, \cdots, z \partial_{\dim H(P)} I)$.

Since $z \partial_i I \equiv e^{i/z} T_i \equiv z \partial_i J$ (mod $q^*$), we have $B \equiv Id$ (mod $q^*$) and this implies the isomorphism on $\mathcal{D}$ modules

$$\mathcal{D}^\tau I(t) \cong \mathcal{D}^\tau J(\tau)$$

up to a generalized mirror transform $\tau(t)$ on $H(X)$. Now it is clear that

$$p(t, z, q^*, z \partial_\bullet) I^Y(t, z, z^{-1}) = J^Y(\tau, z^{-1})$$

for some linear operator $p$. In fact this operator plays the role to remove the $z^{\geq 0}$ terms in $I^Y$ and it can be effectively constructed by induction on $NE(P)$ (cf. [12, Theorem 1.10] for a related construction). The map $t \mapsto \tau(t)$ is then determined by matching the $1/z$ coefficients on both sides.

We also notice that from (2.1) the matrix $B^{-1}$ is the gauge transformation to bring the wrong frame $z \partial_i I$’s back to the preferred frame $z \partial_i J$ so that the connection matrix takes the form expected in (1.1).
3. QUANTUM LERAY–HIRSCH

3.1. Factorial trick for split projective (toric) bundles. Let

\[ \pi : P = P_X(V) \to X \]

be a projective bundle. The classical Leray–Hirsch theorem computes the cohomology of the total space \( P \) in terms of the base \( X \) and the fibers. Let \( h = c_1(\mathcal{O}_P(1)) \), then

\[ H(P) \cong \pi^*H(X)[h]/(f_V(h)) \]

where \( f_V(h) \) is the Chern polynomial of the vector bundle \( V \to X \).

It is natural to ask for a similar description on quantum cohomology. For this purpose we assume that \( \text{QH}(X) \) is given, and \( V = \bigoplus_{i=1}^r L_i \) is a sum of line bundles. We seek for an analogous factorial trick as in the case of quantum Lefschetz. However the formulation must be different since now \( \text{QH}(P) \) contains additional variables.

Let \( \bar{t} \in H(X) \) be a general element from the base, \( D = t^h \) be the fiber divisor class with coordinate \( t^h \), and we consider the mixed variable \( \hat{t} = \bar{t} + D \).

Then a hypergeometric modification of \( J^X \) is defined by

\[ I_P(\hat{t}, z, z^{-1}) = \sum_{\beta \in \text{NE}(P)} q^\beta J^X_{\pi, \beta}(\hat{t}) \times e^{D \cdot (D, \beta)} \prod_{i=1}^r \frac{1}{(h + L_i)_{\beta}}. \]

Here the convention on factorial \( \prod^\Lambda := \prod^\Lambda / \prod^\Lambda \) is used so that \( 1/(L)_{\beta} \) makes sense even if \( L, \beta < 0 \).

When \( X = \text{pt} \), this is the \( I \) function of \( P^{r-1} \) coming from localizations. In general (3.1) arises from fiber localization, which exists by the split assumption on \( V \). The formulation works for other fiber bundles as long as the fiber localization is well understood. In that case \( D = \sum_{i=1}^r t^iD_i \) for \( D_1, \ldots, D_r \) being a basis of \( H^2(P/X) \). The following is due to Brown:

**Theorem 3.1.** [1] Given \( J^X(\bar{t}) \) in \( \bar{t} \in H(X) \), then \( I^P \in \mathcal{O}^P \). More precisely, there is a linear differential operator \( b = \sum_{\beta \in \text{NE}(P)} q^\beta b_\beta \) with \( \text{deg}_z b_\beta < \infty \), and a graph \( \hat{t} \mapsto \tau(\hat{t}) : H(X) \oplus \text{Ch} \to H(P) \), such that

\[ I^P(\bar{t}, z, z^{-1}) = b(\tau, z, q^\bullet, z\partial_\bullet) I^P(\tau, z^{-1}). \]

**Remark 3.2.** The result was proved in [1] for split toric bundles, and stated in the language of Lagrangian cones. We have presented it in an equivalent form to avoid introducing this machinery.

3.2. The Dubrovin connection. Based on Theorem 3.1, Lee, Lin and Wang had developed a method to compute the Dubrovin connection on the bundle space \( P \) [12]. It is roughly represented by following implication:

\[ \text{PF}^{P/X} + \nabla^X \Longrightarrow \nabla^P. \]
To be precise, we need to introduce a system of equations controlling both the fiber directions and the base directions.

For the fibers, we introduce the Picard–Fuchs ideal: For the primitive fiber curve class $\ell \in NE(P/X)$, it is easily checked that $\square_{\ell} I = 0$ where

$$\square_{\ell} = \prod_{i=1}^{\ell} z \partial h_{+L_i} - q^\ell e^{th}. \quad (\partial_L \text{ is the directional derivative in direction } L.)$$

For differentiations in the base variables, we introduce $\hat{\partial}$.

As in §2.2, we have the Birkhoff factorization matrix $B$ such that

$$(\partial^{\omega} I)(\hat{\ell}, z, z^{-1}) = (z \hat{\partial} J)(\tau, z^{-1}) B(\tau, z).$$

In fact $B^{-1}$ is the gauge transformation to remove $z^{20}$ in $C_a(\hat{\ell}, z)$. Furthermore, the map $\hat{\ell} \rightarrow \tau(\hat{\ell})$ is uniquely determined by matching the 1/z coefficients of the first column of $(\partial^{\omega} I)B^{-1}$ with $J$:

$$J(\tau, z^{-1}) = z \partial_1 J = p(\hat{\ell}, z, \partial^{\omega}) I(\hat{\ell}, z, z^{-1}).$$
Set \( z = 0 \) in the gauge transformation we find \(- (z \partial_z B) B^{-1} \mapsto 0\) and

\[
B_0 \, C_{a;0} \, B_0^{-1}(\hat{t}) = \sum_{i=1}^{\dim \, H(P)} A_i(\tau(\hat{t})) \frac{\partial \tau^i}{\partial \hat{t}^a}(\hat{t}),
\]

where \( B_0 = B(z = 0) \) and \( C_{a;0} = C_a(z = 0) \).

Since \( \tau \equiv \hat{t} \mod q^* \), by the Mori cone induction and divisorial reconstruction we may then determine all the Dubrovin connection matrices \( A_i(\hat{t}) \) from (3.4). Of course the computations involved are necessarily complicate and very demanding. In applying these results special attention is paid to maintain the structural information. We will demonstrate on this through a few applications.

4. Application I: Ordinary flops

4.1. The statement. Let \( f : X \to X' \) be a \( P^r \) flop. That is, there are two vector bundles \( F, F' \to S \) of the same rank \( r \), such that the exceptional loci \( Z \subset X \) has the following projective bundle structure:

\[
\text{Exc} \, f = Z = P_S(F) \xrightarrow{\varphi} S.
\]

Moreover, the normal bundle of \( Z \) in \( X \) is given by

\[
N = N_Z = \varphi^* F' \otimes \mathcal{O}_Z(-1).
\]

It was shown in [10] that the graph correspondence \( \mathcal{T} = [\Gamma_f] \) induces an isomorphism on cohomology spaces \( H(X) \cong H(X') \), but it does not preserve the effectivity of one cycles. Indeed if \( \ell \) (resp. \( \ell' \)) is the class of extremal ray in \( X \) (resp. \( X' \)) then

\[
\mathcal{T} \ell = -\ell'.
\]

Moreover, \( \mathcal{T} \) preserves the Poincaré pairing, but not the product structure. It turns out that the topological defects are corrected by the extremal ray GW invariants and the following is true:

**Theorem 4.1.** [10, 11, 12, 9] The graph correspondence \( \mathcal{T} \) induces isomorphism of big quantum cohomology rings \( \mathcal{QH}(X) \cong \mathcal{QH}(X') \) under the analytic continuations induced from \( q^\beta \mapsto q^{\mathcal{T} \beta} \).

Here is a brief history on this problem.

For \( \dim X = 3 \), the multiple cover formula for \( \mathcal{O}_{P^1}(-1)^2 \to P^1 \) gives the quantum corrections of \( (\mathcal{TD})^3 - D^3 \) (cf. Witten [24]). The global case was treated by Li–Ruan [18] around 2000. They proved a degeneration formula of GW invariants in the symplectic category, and used it to show that in fact no degeneration might occur in the threefold case. The problem was then reduced to the case of extremal rays which had already been solved.

In higher dimensions the statement was conjectured to hold for general birational \( K \)-equivalent manifolds in [22]. The case of simple ordinary flops, namely \( S = \text{pt} \), was solved by the LLW team in 2006 [10]. We worked in the
algebraic category and reduced the problem to local models by way of the deformations to the normal cone and the degeneration formula of Li [17]. In this case non-trivial degenerations do arise and the relations between relative GW invariants, descendent invariants, and absolute GW invariants are carefully studied through degenerations (inspired by a method of Maulik and Pandharipande in [20]). For local models, \( X = P_{\mathcal{O}(1)}^{\oplus (r+1)} \) is a semi-Fano toric variety whose GW theory is well studied (cf. Theorem 2.1). In fact \( I^X = J^X \) on small parameters, and the analytic continuation can be solved. The result was further extended to the higher genus GW theory in [7] by studying ancestor invariants and quantization.

For general base \( S \) with split bundles \( F, F' \), the analytic continuation was later solved by LLW in 2011. Indeed, the problem was reduced to the local models in [11], and the case of local models was solved through the quantum Leary–Hirsch theorem in [12]. More recently, a quantum splitting principle is proved by Lee, Lin, Qu and Wang in [9], which reduced the problem for general vector bundles \( F, F' \) to the case of split bundles, hence proved Theorem 4.1 completely.

4.2. A sketch of proof. Now we sketch how the Quantum Leray–Hirsch is applied to solve the case of local split flops.

The flop is achieved by first blowing up \( Z \subset X \) to get \( Y = \text{Bl}_Z X \to X \) and then contracting the exceptional divisor \( E = P_{\mathcal{O}(1)} \times S P_{\mathcal{O}(1)} \subset Y \) in another fiber direction to get \( Z' \subset X' \). We have \( \bar{\psi}' : Z' = P_{\mathcal{O}(1)} \to S \) being a projective bundle and \( N' = N_{Z'} = \bar{\psi}'^* F \otimes \mathcal{O}_{Z'}(-1) \):

\[
X = P_Z (N \oplus \mathcal{O}) \xrightarrow{f} X' = P_{Z'} (N' \oplus \mathcal{O}) \xleftarrow{\bar{\psi}'} S \xrightarrow{\bar{\psi}} \]

As a double projective bundle \( X \xrightarrow{\pi} Z \xleftarrow{\bar{\psi}} S \), we have \( NE(X/S) = \langle \ell, \gamma \rangle \), where \( \ell \) (resp. \( \gamma \)) is the \( \bar{\psi} \) (resp. \( \pi \)) fiber line classe. Let \( \xi = \mathcal{O}_X(1) \) and \( h = \mathcal{O}_Z(1) \), and \( D = t^h h + t^\xi \xi \in H^2(X/S) \) a general fiber divisor. Then

\[
H(X) = \mathbb{Z}[h, \xi] / (f_F, f_{N\oplus \mathcal{O}}),
\]

where \( f_V \) is the Chern polynomial of a bundle \( V \).

When \( F = \bigoplus_{i=1}^r L_i, F' = \bigoplus_{i=1}^r L'_i \) are split bundles, we have

\[
f_F(h) = \prod (h + L_i),
\]

\[
f_{N\oplus \mathcal{O}}(h, \xi) = \xi \prod (\xi - h + L'_i).
\]

By symmetry we have similar formulae on the \( X' \) side. However, it is not compatible with the \( X \) since \( \mathcal{T} h = \xi' - h', \mathcal{T} \xi = \xi' \). Thus the cup product structure is not preserved under \( \mathcal{T} \). (See [11, Theorem 1.8] for the explicit computations on the topological defects.)
Now comes the key point: To remedy the topological defects we replace the cohomology class by its “quantized” version, namely we consider differential operators instead. This gives rise to the Picard–Fuchs operators

$$\Box_\ell = \prod z \partial_{h + L_i} - q^\ell e^{\ell h} \prod z \partial_{e^{-h + L_i} \ell}$$
$$\Box_\gamma = z \partial_z \prod z \partial_{e^{-h + L_i} \ell} - q^\gamma e^{\gamma \ell} \prod z \partial_{e^{\gamma h} - L_i}$$

which are regarded as the “quantized version” of the Chern polynomials. Similarly we have on the $X'$ side

$$\Box_{\ell'} = \prod z \partial_{h' + L_i} - q^\ell e^{\ell h'} \prod z \partial_{e^{h' - h + L_i} \ell'}$$
$$\Box_{\gamma'} = z \partial_z \prod z \partial_{e^{h' - h + L_i} \ell'} - q^\gamma e^{\gamma \ell'} \prod z \partial_{e^{\gamma h'} - L_i}.$$

The coordinates are related by requiring $t^h h' + t^\xi \xi' = T(t^h h + t^\xi \xi) = t^h (\xi' - h') + t^\xi \xi'$. That is, $t^h = -t^h$ and $t^\xi = t^\xi + t^h$.

Now it is a simple exercise to check that

**Lemma 4.2.** $T$ induces an isomorphism on Picard–Fuchs ideals

$T \langle \Box_\ell, \Box_\gamma \rangle \cong \langle \Box_{\ell'}, \Box_{\gamma'} \rangle$.

**Remark 4.3.** Lemma 4.2 can be extended to split toric bundle flops.

For the base directions, the lift of QDE in (3.3) is independent of the choice of admissible lift $\beta^*$ modulo the Picard–Fuchs ideal. The admissible condition for $\beta$ in this case is given by $-\beta(h + L_i) \geq 0$, $-\beta(\xi - h + L_i) \geq 0$ and $-\beta \xi \geq 0$. It is readily seen that $\beta$ is admissible in $X$ if and only if $T \beta$ is admissible in $X'$. This implies that the lifting of QDE from $S$ to $X$ and the one from $S$ to $X'$ are indeed equivalent under $T$ modulo the Picard–Fuchs ideal. Thus, the quantum Leray–Hirsch theorem implies that $X$ and $X'$ have compatible first order PDE systems up to analytic continuations.

To achieve $T : QH(X) \cong QH(X')$, we still need to show that the Birkhoff factorization $B$ and the generalized mirror map $\tau(t)$, as appeared in (3.4), are compatible on both sides. We refer the details to [12, §3.3].

### 5. Application II: Blow-ups along c.i. centers

Let $L_i = \mathcal{O}_X(D_i)$, $1 \leq i \leq r$, and $Z = D_1 \cap \cdots \cap D_r$ a smooth complete intersection subvariety of $X$ with codimension $r$. Let $\mathcal{E} = \oplus_{i=1}^r L_i$ with a given section $s = (s_i)$ such that $D_i = (s_i)$, $Z = s^{-1}(0)$. Consider the blow-up $\phi : Y \to X$ along $Z$:

$$Z \hookrightarrow E \twoheadrightarrow Y = \text{Bl}_Z X.$$
By the construction, we have a surjective morphism $\mathcal{E}^* \to I_Z$. This leads to the embedding

$$\iota : Y := \text{Proj}_X \bigoplus_{d=0}^\infty \mathcal{I}_d^d \hookrightarrow \text{Proj}_X \text{Sym } \mathcal{E}^* = P_X(\mathcal{E}) .$$

Let $\pi : P := P_X(\mathcal{E}) \to X$ be the bundle map with associated Euler sequence $0 \to S \to \pi^* \mathcal{E} \to Q \to 0$ over $P$. It is shown in [14] that there is a canonical section $\sigma \in \Gamma(P, Q)$ of the universal quotient bundle such that $Y = \sigma^{-1}(0) \subset P$. We emphasize that this is not true if $Z$ is not a complete intersection. The situation is summarized in the following diagram:

(5.1) $$\begin{array}{cccccc}
0 & \longrightarrow & S & \longrightarrow & \pi^* \mathcal{E} & \longrightarrow & Q & \longrightarrow & 0 .
\end{array}$$

Let $\eta = c_1(\mathcal{O}_P(1))$. It follows that $S = \mathcal{O}_P(-\eta)$ and $-\eta|_Y = E$ (cf. [5]).

Notice that the quotient bundle $Q$ is in general not a split bundle over $P$ and the quantum Lefschetz cannot be applied directly. However, a suitable extension of it to short exact sequences allows us to apply it to the Euler sequence (5.1) and now both $S$ and $\pi^* \mathcal{E}$ are split bundles.

With the above understood, the (extended) quantum Lefschetz together with the quantum Leray–Hirsch lead to the following factorial trick on the $\beta$ component of $I_Y$:

$$I_Y^\beta = I^P_\beta \frac{\prod_{i=1}^r (D_i)_{\beta}}{(-\eta)_\beta} \sim \int_{\tau, \beta}^{X} e^{r\eta} \frac{\prod_{i=1}^r (D_i)_{\beta}}{(-\eta)_\beta \prod_{i=1}^r (\eta + D_i)_{\beta}} .$$

(5.2)

Here $t^\phi$ denotes a suitable dual coordinate of $\eta$.

**Theorem 5.1.** [14] For the smooth blow-up $\phi : Y \to X$ along a complete intersection center $Z = \bigcap_{i=1}^r D_i$, the relative I factor is given by

$$I_{Y/X} = e^{s(E + E, \beta)} \left( \prod_{i=1}^r \frac{(D_i)_{\beta}}{(D_i - E)_{\beta}(E)_{\beta}} \right) (E)^{r-1} ,$$

where $E \subset Y$ is the exceptional divisor and $s$ is a suitable dual coordinate.

**Remark 5.2.** The formula suggests nice structures of the relative factor. Indeed, $K_Y = \phi^* K_X + (r - 1)E$ and $(E)^{r-1}$ is responsible for the Jacobian of $\phi$. For each $i$, $(D_i)_{\beta}/((D_i - E)_{\beta}(E)_{\beta})$ is a Calabi–Yau factor which describes the decomposition of the linear system $|D_i - Z|$ into moving part and the
fixed part (on Y). We expect that this intrinsic formulation will be useful for general blow-ups.

As in quantum Leray–Hirsch described in §3.2, to get the Dubrovin connection on Y (or QH(Y)) we proceed by (1) choosing the corresponding naïve quantization basis (2) determining the Picard–Fuchs ideal on fibers (3) finding the lifting of QDE on the base X to Y. Then (1) + (2) + (3) \( \implies \mathcal{Q} \mathcal{F} Y \implies QH(Y) \). The details will appear in [14].

6. Application III: Simple Flips

Let \( r, r' \in \mathbb{N} \) and \( r > r' \). In the definition of ordinary flops, if the underlying vector bundles \( F \) and \( F' \) have different rank \( r \) and \( r' \) respectively then in exactly the same construction as §4.2 we get ordinary \( (r, r') \) flips. The effect on quantum cohomology under flips are discussed in [15]. In contrast to analytic continuations in the flops case, the situation for flips is more complicated and new phenomenon appears. We give a sketch of our results in the simplest case of local models of simple \((2, 1)\) flips.

6.1. \( H(X) \) vs \( H(X') \) and the Picard–Fuchs systems. The local model of \((2, 1)\) flips has the following data\( L = P^2, Z' = P^1, \) and

\[
 f : X = P_{p^2}(\mathcal{O}(-1)^2 \oplus \mathcal{O}) \longrightarrow X' = P_{p^1}(\mathcal{O}(-1)^3 \oplus \mathcal{O})
\]

The cohomology rings are given by

\[
 H(X) = \mathbb{Z}[h, \xi]/(h^3, \xi(\xi - h)^2),
\]

\[
 H(X') = \mathbb{Z}[h', \xi']//(h'^2, \xi'(\xi' - h')^3),
\]

where \( \dim H(X) = 9 \) and \( \dim H(X') = 8. \)

The graph correspondence \( \mathcal{G} = [\Gamma_f] \), induces a short exact sequence

\[
 0 \longrightarrow K \longrightarrow H(X) \xrightarrow{\mathcal{G}} H(X') \longrightarrow 0
\]

where \( K = \ker \mathcal{G} = Ck_1 \) with \( k_1 = (\xi - h)^2 = [Z]. \)

The transpose correspondence \( \mathcal{G}^{-1} := [\Gamma_{f}] \), preserves the Poincaré pairing and induces an imbedding \( \mathcal{G}^{-1} : H(X') \hookrightarrow H(X) \) (indeed, of motives) which leads to an orthogonal decomposition (cf. [10, §2.3])

\[
 H(X) = \mathcal{G}^{-1}H(X') \mathcal{H} K.
\]

However, \( \mathcal{G}^{-1} \) does not preserves the cup product. In fact \( K \mathcal{H} \) is not closed under cup product. As in the case of flops we have curve classes \( \ell, \gamma \) in \( X \) and \( \ell', \gamma' \) in \( X' \). They are related by \( \mathcal{G}\ell = -\ell' \) and \( \mathcal{G}\gamma = \ell' + \gamma' \). Also \( \mathcal{G}h = \xi' - h' \) and \( \mathcal{G}\xi = \xi'. \)

The divisor variable takes the form \( D = t^1h + t^2\xi. \) To simplify notations in our discussion, we will use variables

\[
 q_1 = q\ell t^1, \quad q_2 = q\gamma t^2; \quad q'_1 = q\ell' e^{-t^1}, \quad q'_2 = q\gamma' e^{t^1+t^2}.
\]
From the computational point of view of quantum cohomology, \( X' \) is bad since \( c_1(X') = -h + 4\xi \) which contains both \( K \) positive and \( K \) negative directions. In other words, the \( I^X \) is complicate and contains positive \( z \) powers. Its Picard–Fuchs equations are given by

\[
\square \ell' = (z \partial_{\ell'})^2 - q_1(z \partial_{\ell'-h'})^2, \quad \square \gamma' = z \partial_{\gamma'}(z \partial_{\gamma'-h'})^3 - q_2.
\]

The first equation \( \square \ell' I^{X'} = 0 \) shows that in order to reduce \( (z \partial_{\ell'}) I^{X'} \) we will receive derivatives of even higher power. It does give the correct reduction algorithm in the Mori cone topology since there is also a \( q_1' \) multiplied. It is difficult to compute \( \nabla^{X'} \) or to get any structure of it form this approach.

On the other hand, \( X \) is toric Fano with \( c_1(X) = h + 3\xi \). The computation of \( QH(X) \) is in principle easy since \( I^X = J^X \) along small parameters. Assume that this has been done, then the natural question is

"Can we get \( QH(X') \) from \( QH(X) \) in a canonical manner?"

Now we restrict ourselves to the small parameters \( t \in H^0 \oplus H^2 \) so that we can work with variables \( q_1, q_2 \) and \( q_1', q_2' \) directly. The Picard–Fuchs equations on \( J := J^X \) can be easily determined to be

\[
\square \ell = (z \partial_{\ell})^3 - q_1(z \partial_{\ell-h})^2, \quad \square \gamma = z \partial_{\gamma}(z \partial_{\gamma-h})^2 - q_2.
\]

It is closely related the one for \( X' \):

**Lemma 6.1.** Along the partially compactified two dimensional Kähler moduli \( \mathcal{K} := \{(q_1, q_2)\} \cup \{(q_1' = 1/q_1, q_2' = q_1q_2)\} \cong \Theta_{p_1}(1) \), we have

\[
\mathcal{F} : \langle \square \ell, \square \gamma \rangle \cong \langle \square \ell', \square \gamma' \rangle
\]

outside the divisors \( D_0 = \{q_1 = 0\} \) and \( D_\infty = \{q_1' = 0\} \)

The isomorphism does not extend over \( D_0 \) or \( D_\infty \) due to the rank defect.

**6.2. Exact formula for \( \nabla^X \) [15].** The following frame (recall that \( I = J \))

\[
\begin{align*}
v_1 &= \hat{I} J = J, \\
v_2 &= \hat{h} J, \quad v_3 = (\xi - \hat{h}) J, \\
v_4 &= \hat{h}^2 J - (\xi - \hat{h})^2 J, \quad v_5 = \hat{h}(\xi - \hat{h}) J + (\xi - \hat{h})^2 J, \\
v_6 &= \hat{h}^3 J - \hat{h}(\xi - \hat{h})^2 J, \quad v_7 = \hat{h}^2(\xi - \hat{h}) J + \hat{h}(\xi - \hat{h})^2 J, \\
v_8 &= \hat{h}^3(\xi - \hat{h}) J + \hat{h}^2(\xi - \hat{h})^2 J, \\
v_9 &= \hat{k}_1 = (\xi - \hat{h})^2 J,
\end{align*}
\]

respects \( H(X) = \mathcal{F}^{-1} H(X') \oplus \perp K \) when modulo \( q_1, q_2 \). They are precisely

\[
z \partial_i J \quad \text{at} \ t \in H^0 \oplus H^2, \quad 1 \leq i \leq 9,
\]
and we get the Dubrovin connection matrices

\[
A_1 = h_{\text{small}} = \begin{pmatrix}
1 & q_1 q_2 \\
1 & q_1 q_2 \\
1 & 1 \\
1 & -1 \\
1 & 1 \\
1 & 1 \\
1 & 1 \\
1 & 1 \\
1 & 1 \\
1 & 1 \\
\end{pmatrix},
\]

\[
A_2 = \xi_{\text{small}} = \begin{pmatrix}
-1 & q_1 q_2 & q_1 q_2 & q_2 \\
-1 & q_1 q_2 & q_2 \\
1 & q_2 \\
1 & 1 \\
1 & 1 \\
1 & 1 \\
1 & 1 \\
1 & 1 \\
1 & 1 \\
1 & 1 \\
\end{pmatrix}.
\]

The crucial observation is that there is a single appearance of \(q_1\) in \(A_{1,9}\). This shows that the system has irregular singularity along \(D_\infty = (q_1 = \infty)\) in the \(K\) direction. Let \(x = q'_1, y = q'_2\). After a constant change of basis from \(v_i's\) to \(w_i's\) such that the Poincaré pairing \((w_i, w_j) = \delta_{9,i+j}, 1 \leq i, j \leq 8\) and \(w_9 := v_9\) with \((w_9, w_i) = \delta_{9,i}, the fundamental solution matrix \(S\) satisfies

\[
z(x \partial_x)S = \begin{pmatrix}
\frac{1}{2} xy & xy \\
1 & \frac{1}{2} xy \\
1 & -\frac{1}{2} xy \\
1 & 1 \\
1 & \frac{1}{2} xy \\
1 & 1 \\
1 & xy \\
1 & -1/xy \\
\end{pmatrix},
\]

which is irregular in the \(K\)-block, i.e. the (9, 9) entry, of Poincaré rank one.

6.3. **Block diagonalization.** The classical theory of ODE and the flatness of \(\nabla_x^X\) imply that there exists a unique formal gauge transformation \(S = PZ:\)

\[
P(x, y, z) = \begin{pmatrix}
1 & g_1 \\
\vdots & \vdots \\
f_1 & \cdots & f_8 \\
\end{pmatrix},
\]

(6.1)
such that
\[ z(x \partial_x)Z = B_1 Z, \quad z(y \partial_y)Z = B_2 Z, \]
with \( B_1, B_2 \) block diagonalized. Moreover, each \( f_i(x, y, z) = -g_{9-i}(x, y, -z) \)
is a formal series expansion of certain special function. The claim is that
we may relate the first \( 8 \times 8 \) blocks of \( B_1(x, y, z) \) and \( B_2(x, y, z) \) with the
Dubrovin connection \( \nabla^{X'} \).

Under the new \( z \)-dependent frame \( \tilde{w}_1, \ldots, \tilde{w}_8, \tilde{k}_1 \) from (6.1), namely
\begin{equation}
(6.2) \quad \tilde{w}_i = w_i + f_i \tilde{k}_1, \quad \tilde{k}_1 = \tilde{k}_1 + \sum_{i=1}^{8} g_i w_i,
\end{equation}
we have (a special case of [15] for \((r, r') = (2, 1)\):

**Theorem 6.2.** For a simple \((2, 1)\) flip \( f : X \dashrightarrow X' \), under the frame (6.2) at
\( z = 0 \), we have a ring isomorphism
\[ QH(X) \cong \langle \tilde{w}_1(0), \ldots, \tilde{w}_8(0) \rangle \times \mathbb{C}. \]
Moreover, \( \langle \tilde{w}_1(0), \ldots, \tilde{w}_8(0) \rangle \cong QH(X') \) as \( \mathcal{D}^\times \) modules, but not as rings.

The proof is based on Lemma 6.1 and we refer to [15] for the details.

### 7. Conifold Transitions of CY 3-folds

#### 7.1. Relations on vanishing \( \Lambda \) and \( B \) cycles.

A Calabi–Yau variety is a \( \mathbb{Q} \)-Gorenstein variety with \( K \sim 0 \) and \( h^1(\mathcal{O}) = 0 \).

Let \( X \not
\rightarrow Y \) be a *projective* conifold transition of Calabi–Yau 3-folds \( X, Y \) through a singular Calabi–Yau variety \( \tilde{X} \) with \( k \) ODPs \( p_1, \ldots, p_k \in \tilde{X} \). During the complex degeneration \( \pi : X \rightarrow \Delta \) with \( \pi_0 = \tilde{X} \), there are \( k \) vanishing 3-spheres \( S_1, \ldots, S_k \) with \( N_{S_i/X} = T^*S^3 \). And during the Kähler degeneration (small contraction) \( \psi : Y \rightarrow \tilde{X} \), there are \( k \) vanishing 2-spheres (exceptional curves) \( C_1, \ldots, C_k \) with \( N_{C_i/Y} = \mathcal{O}_{p_i}(-1)^{\oplus 2} \):

\[ C_i \subset Y. \]

\[ S_i \subset X \xrightarrow{\pi} p_i \in \tilde{X} \]

Let \( \mu := h^{2,1}(X) - h^{2,1}(Y) > 0 \) be the lose of complex moduli and \( \rho := h^{1,1}(Y) - h^{1,1}(X) > 0 \) be the gain of Kähler moduli. From \( \chi(X) - k\chi(S^3) = \chi(Y) - k\chi(S^2) \), we get the following well-known elementary relation
\[ \mu + \rho = k. \]

This implies that the \( \psi \)-exceptional curve classes \( [C_i] \in NE(Y/\tilde{X}) \) admit \( \mu \)
independent relations, and the \( \pi \) vanishing cycles \( [S_i] \in V \hookrightarrow H_3(X) \to \)
$H^3(\bar{X})$ admit $\rho$ independent relations. (The vanishing cycle space $V$ has $\dim V = \mu$.) Let $A, B$ be the corresponding relation matrices:

$$A = (a_{ij}) \in M_{k \times \mu}, \quad \sum_{i=1}^{k} a_{ij} [C_i] = 0,$$

$$B = (b_{ij}) \in M_{k \times \rho}, \quad \sum_{i=1}^{k} b_{ij} [S_i] = 0.$$

**Theorem 7.1** (Basic exact sequence) [13, Theorem 1.14] The Hodge realization of $\mu + \rho = k$ is represented by an exact sequence

$$0 \to H^2(Y) / H^2(X) \xrightarrow{B} C^k \xrightarrow{A^t} V \to 0$$

of weight two Hodge structures.

Indeed $V \cong H^3_{\infty}(X)$ in the limiting Hodge diamond for $\pi$:

![Hodge Diamond Diagram]

and the invariant subsystem is $\text{Gr}^{\infty}_{\mu} H^3(X) \cong H^3(Y)$.

7.2. **Local quantum transition.** By the Bogomolov–Tian–Todorov theorem and its extension to Calabi–Yau conifolds by Ran and Kawamata, the moduli spaces $\mathcal{M}_Y$ and $\mathcal{M}_{\bar{X}}$ are smooth of dimension $h^{2,1}(Y)$ and $h^{2,1}(X)$ respectively. Also the contraction $\psi: Y \to \bar{X}$ deforms in projective families. This then identifies $\mathcal{M}_Y$ as a codimension $\mu$ boundary strata in $\mathcal{M}_{\bar{X}}$ and locally near $[\bar{X}] \in \mathcal{M}_{\bar{X}}$ we have $\mathcal{M}_{\bar{X}} \cong \Delta^\mu \times \mathcal{M}_Y$.

We represent $V = C(\Gamma_1, \ldots, \Gamma_\mu)$ in terms of a basis $\Gamma_j$'s. It was shown in [13, Proposition 3.15] that the $\alpha$-periods

$$r_j = \int_{\Gamma_j} \Omega, \quad 1 \leq j \leq \mu$$

form the degeneration coordinates around $[\bar{X}] \in \mathcal{M}_{\bar{X}} \cong \Delta^\mu \times \mathcal{M}_Y$.

In order to describe the discriminant loci of $M_\bar{X}$ near $[\bar{X}]$, we recall Friedman’s result on (partial) smoothing of ODP’s:

**Proposition 7.2.** [4] Let $w_i = a_{i1} r_1 + \cdots + a_{i\mu} r_\mu$, then the divisor $D_i := \{ w_i = 0 \} \subset \mathcal{M}_{\bar{X}}$ is the loci where the sphere $S_i$ shrinks to an ODP $p_i$.

It is clear that the discriminant loci $D_B = \bigcup_{j=1}^{k} D_j$ is not a normal crossing divisor. Rather it is a central hyperplane arrangement.
Under a suitable choice of homology symplectic basis, the $\beta$-periods in the transversal directions are given by

$$u_p = \partial_p u = \int_{\beta_p} \Omega$$

for some function $u$. The Bryant–Griffiths–Yukawa couplings are then extended over the boundary $D_B$ and satisfy

$$u_{pmn} := \partial^3_{pmn} u = O(1) + \sum_{i=1}^k \frac{1}{2\pi \sqrt{-1}} \frac{a_{ip} a_{im} a_{in}}{w_i}$$

for $1 \leq p, m, n \leq \mu$. It is regular if one of the indices is outside this range.

The collection $\{u_{pmn}\}$ is the essential part of the Gauss–Manin connection $\nabla^{GM}$ on $\mathcal{M}$ which has regular singular extension over $D_B$.

Similarly, let $u = \sum_{p=1}^\mu u^T_p \in H^2(Y/X)$, $D^i := \{\sum_{p=1}^\mu b_{ip} u^p = 0\}$, $i = 1, \ldots, k$. By the multiple cover formula of GW invariants we know that $QH(Y)$ is regular singular along $D_A = \bigcup D^i$.

Let $y = \sum_{i=1}^k y_i e_i \in C^k$, with $e^1, \ldots, e^k$ being the dual basis on $(C^k)$. The trivial logarithmic connection on $C^k \oplus (C^k)^\vee \rightarrow C^k$ is defined by

$$\nabla^k = d + \frac{1}{z} \sum_{i=1}^k \frac{dy_i}{y_i} \otimes (e^i \otimes e^*_i).$$

The statement $A^TB = 0$ in Theorem 7.1 leads to an orthogonal sum

$$(7.1) \quad C^k = \text{image } A \oplus \text{image } B \cong V^* \oplus H^2(Y)/H^2(X).$$

**Theorem 7.3.** [13, Theorem 4.1] Under the identification (7.1),

1. $\nabla^k$ restricts to the logarithmic part of $\nabla^{GM}$ on $V^*$.
2. $\nabla^k$ restricts to the logarithmic part of $\nabla^{Dubrovin}$ on $H^2(Y)/H^2(X)$.

**7.3. Global aspects.** Denote by $\mathcal{A}(-)$ the GW theory and $\mathcal{B}(-)$ the variations of Hodge structure. Theorem 7.3 provides evidence to

"excess $\mathcal{A}$ theory" + "excess $\mathcal{B}$ theory" = "trivial"

through the partial exchange of quantum information attached to vanishing cycles on both the $\mathcal{A}$ and $\mathcal{B}$ theories. For the full information on quantum $\mathcal{A}$, $\mathcal{B}$ theories, we proved the following result:

**Theorem 7.4.** [13, Theorem 0.3] Let $[X]$ be a nearby point of $[\bar{X}]$ in $\mathcal{M}_X$,

1. $\mathcal{A}(X)$ is a sub-theory of $\mathcal{A}(Y)$ (e.g. quantum sub-ring in genus 0).
2. $\mathcal{B}(Y)$ is a sub-theory of $\mathcal{B}(X)$ (invariant sub-VHS).
3. $\mathcal{A}(Y)$ can be reconstructed from a "refined $\mathcal{A}$ theory" on

$$X^\circ := X \setminus \bigcup_{i=1}^k S_i$$

"linked" by the vanishing spheres in $\mathcal{B}(X)$.
(4) $\mathcal{A}(X)$ can be reconstructed from the VMHS on $H^3(Y^\circ)$,

$$Y^\circ := Y \setminus \bigcup_{i=1}^k C_i,$$

“linked” by the exceptional curves in $\mathcal{A}(Y)$.

The definition of the linked GW invariant in (3) is really a reformulation of the discreteness of components appearing in the virtual cycle form of the degeneration formula for conifold transitions of Calabi–Yau 3-folds:

$$\langle - \rangle_{S,\beta}^{X} = \sum_{c} \langle - \rangle_{S,\gamma^c}^{Y},$$

The sum is a finite sum. However, no method is known to single out the individual term in it. To get $QH(Y)$ from $QH(X)$, it requires a blow-up formula of GW invariants where the blow-up center is a Weil divisor.

Weil divisors on $\bar{X}$ can be constructed from the relation matrix $B$ on $X$.

Indeed, $\sum_{i=1}^k b_{ij}[S_i] = 0$ implies that there are real 4-chain $W_j$, $t$ such that $\sum_{i=1}^k b_{ij}S_i = \partial W_j$, $t$.

When $t \to 0$, we get homology cycles $W_j := W_{j,0}$, $j = 1, \ldots, \rho$, since now $\partial W_j$ is supported at the ODPs. From our definition of Calabi–Yau varieties it can be shown that $W_j$’s are represented by algebraic cycles, hence they give rise to Weil divisors on $\bar{X}$.

The projective small resolution $\psi : Y \to \bar{X}$ transforms all these $W_j$’s into Cartier divisors. Indeed, let $W$ be the sum of supports of all $W_j$’s. Then $Y = Bl_W \bar{X}$, i.e. the blow-up of the ideal sheaf $\mathcal{I}_W \subset \mathcal{O}_{\bar{X}}$. A non-Cartier Weil divisor is simply a non-complete intersection divisor. Thus the problem is essentially a problem on finding a blow-up formula of GW theory with non-complete intersection center. Notice that the GW theory on $\bar{X}$ is so far undefined in the literature. But from the deformation invariance of GW theory we may in practice identify it with the GW theory on $X$.

Example 7.5 (Determinantal transitions). [14] Let $Y \subset S \times P^n$ be the zero loci of sections $s_i \in \Gamma(S \times P^n, \mathcal{L}_i)$ where $\mathcal{L}_i \to S \times P^n$ are line bundles of the form $\mathcal{L}_i = L_i \boxtimes \mathcal{O}_{P^n}(1)$ with $L_i$ being semi-ample on $S$.

Let $[x_0 : \cdots : x_n]$ be the homogeneous coordinates on $P^n$. We write

$$s_i = \sum_{j=0}^n s_{ij} x_j, \quad i = 0, \ldots, n,$$

where $s_{ij} \in \Gamma(S, L_i)$. We are interested in study the restriction of the projection map $\pi : S \times P^n \to S$ to $Y$. Define $\bar{X} = \pi(Y) \subset S$ and

$$\psi = \pi|_Y : Y \to \bar{X}.$$

For $p \in \bar{X} \subset S$, since $s_{ij}(p)$’s are fixed, $\psi^{-1}(p)$ is not unique if and only if

$$\Delta(p) := \det s_{ij}(p) = 0.$$
The contraction $\psi : Y \to \bar{X}$ is called a determinantal contraction, and the variety $\bar{X}$ is defined by equations $\Delta = 0$ on $S$. Notice that

$$\Delta \in \Gamma(S, \bigotimes_{i=0}^{n} L_i).$$

If for general sections $\tau \in \bigotimes_{i=0}^{n} L_i$ the variety $X_\tau$ defined by $\tau = 0$ is smooth, then it gives rise to a transition $Y \searrow X$. If furthermore $\bar{X}$ has only ODPs, then we get a conifold transition. These properties hold CICY 3-folds transitions which have been studied extensively in the literature.

Given a determinantal transition, we proceed to determine the GW theory on $Y$ in terms of the one on $X$ and the data $L_i$'s. Our goal is to replace the extrinsic data $L_i$'s by the intrinsic data associated to $\psi$, namely the Weil divisor $W$ which gives $Y = \text{Bl}_W \bar{X}$. Here we consider the $g = 0$ case:

Let $h = c_1(\mathcal{O}_{\mathbb{P}^n}(1))$. As in (5.2), the quantum Lefschetz gives

$$I_Y = \int S \prod_{i=0}^{n} \frac{(L_i + h)}{(h)^{n+1}},$$

$$I_X = \int S (\sum_{i=0}^{n} L_i),$$

where we omit the curve class $\beta$ in the subscript. Then

$$(7.2) \quad I_{Y/X} = \prod_{i=0}^{n} \frac{(L_i + h)}{(h)^{n+1}(\sum_{i=0}^{n} L_i)} = \prod_{i=0}^{n} \frac{(L_i + h)}{(L_i)(h)} \times \prod_{i=0}^{n} \frac{(L_i)}{(\sum_{i=0}^{n} L_i)}.$$

The question is whether we may write

$$(7.3) \quad \prod_{i=0}^{n} \frac{(L_i + h)}{(L_i)(h)} \sim \prod_{i=0}^{n} \frac{(L_i)}{(L_i - h)(h)}?$$

We note that the divisor $h$ on $S \times \mathbb{P}^n$ coming from $\mathcal{O}_{\mathbb{P}^n}(1)$ restrict to a divisor, still called $h$, on $Y$. If $p \in \bar{X}$ is a point with positive dimensional fiber $\psi^{-1}(p) \subset \{p\} \times \mathbb{P}^n$. Then $h$ intersects $\psi^{-1}(p)$ non-trivially since $h$ comes from a hyperplane in $\mathbb{P}^n$. This effective divisor $W = \psi_* (h) \subset \bar{X}$ is thus the Weil divisor we are seeking for. When $\psi$ is a small contraction, it is then the blowup of $\bar{X}$ along $W$.

From this viewpoint, the validity of (7.3) leads to intrinsic meaning of the $I_{Y/X}$ factor in terms of decompositions of linear systems (as in the case of smooth blow-ups along complete intersection centers, cf. Remark 5.2). The detailed discussions will appear in [14].

References


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