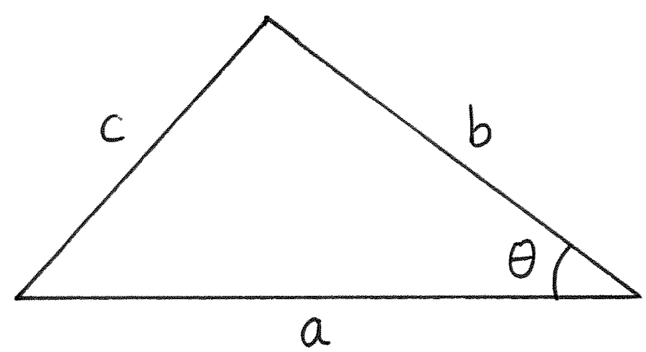
Law of Cosines

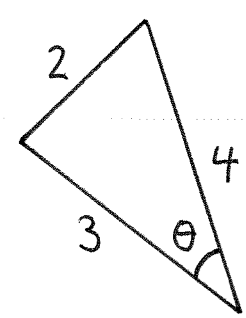
Suppose we have a triangle with one of its angles, θ, identified. Suppose further that the length of the side of the triangle that is opposite the angle θ is c. The other two sides of the triangle have length a and b.



Then the *law of cosines* is the formula

$$c^2 = a^2 + b^2 - 2ab\cos(\theta)$$

Problem. Find $\cos(\theta)$ if θ is the angle shown in the triangle below.



Solution. The law of cosines tells us that

$$2^2 = 3^2 + 4^2 - 2(3)(4)\cos(\theta)$$

Simplified, we have

$$4 = 25 - 24\cos(\theta)$$

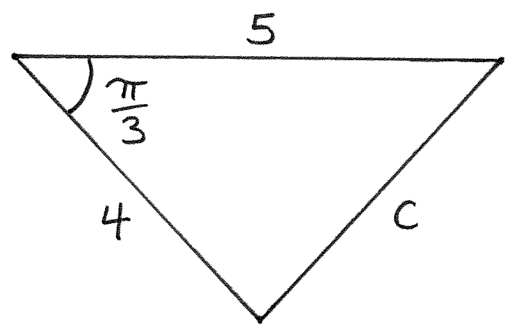
We can subtract 25

$$-21 = -24\cos(\theta)$$

and then divide by -24

$$\cos(\theta) = \frac{-21}{-24} = \frac{7}{8}$$

Problem. Find c if c is the length of the side of the triangle shown below.



Solution. The law of cosines states that

$$c^2 = 4^2 + 5^2 - 2(4)(5)\cos\left(\frac{\pi}{3}\right)$$

Simplified, we have

$$c^2 = 16 + 25 - 40\cos\left(\frac{\pi}{3}\right)$$

We know that $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$ so that the equation can be simplified further as

$$c^2 = 16 + 25 - 20 = 21$$

Therefore, either $c = \sqrt{21}$ or $c = -\sqrt{21}$. However, c is a length, which must be a positive number, so $c = \sqrt{21}$. That's the end of the solution.

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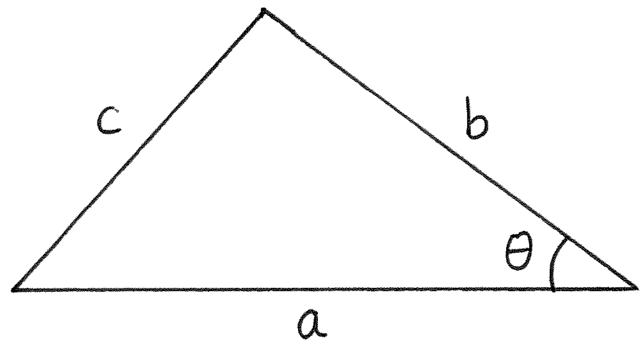
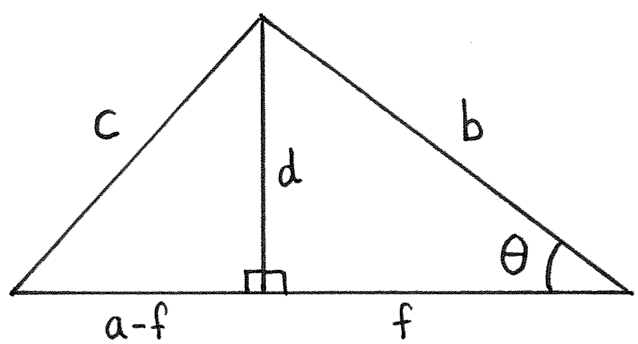
Why the law of cosines is true

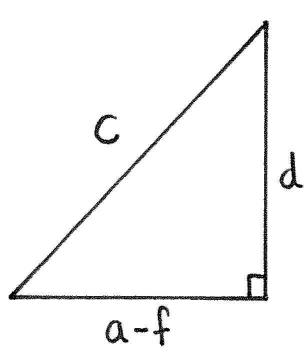
If we have the triangle

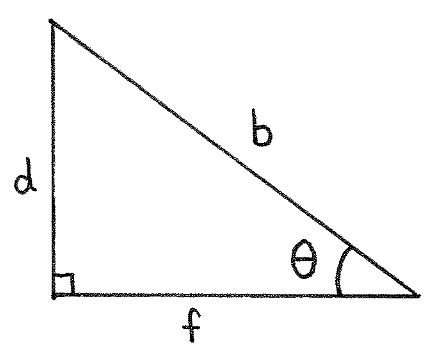


then we can draw another line from the top of the triangle to the base in a way that creates a right angle. We'll say that the length of this new line is d, and that the length of the base to the right of this new line is f. Because the length of the entire base of our original triangle was a, the length of the base to the left of the new line we drew must be $a - f$.



The line we drew cuts our original triangle into two right triangles.





From the right triangle on the right we see that $\sin(\theta) = \frac{d}{b}$ (the length of the side opposite θ divided by the length of the hypotenuse) and that $\cos(\theta) = \frac{f}{b}$ (the length of the side adjacent to θ divided by the length of the hypotenuse).

These last two equations are equivalent to the equations $b\sin(\theta) = d$ and $b\cos(\theta) = f$, respectively. We'll use these two equations in applying the Pythagorean theorem to the triangle on the left:

$$\begin{aligned}
c^2 &= (a-f)^2 + d^2 \\
&= a^2 - 2af + f^2 + d^2 \\
&= a^2 - 2a\big[b\cos(\theta)\big] + \big[b\cos(\theta)\big]^2 + \big[b\sin(\theta)\big]^2 \\
&= a^2 - 2ab\cos(\theta) + b^2\cos(\theta)^2 + b^2\sin(\theta)^2 \\
&= a^2 - 2ab\cos(\theta) + b^2\big[\cos(\theta)^2 + \sin(\theta)^2\big]
\end{aligned}$$

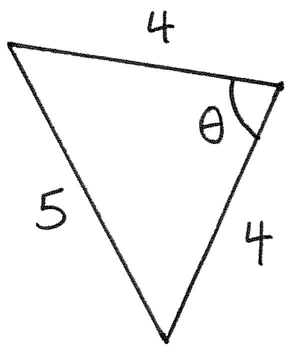
We know that $\cos(\theta)^2 + \sin(\theta)^2 = 1$. This is the Pythagorean Identity. It was one of the identities that we learned in the Sine and Cosine chapter as Lemma 7. Therefore,

$$\begin{aligned}
c^2 &= a^2 - 2ab\cos(\theta) + b^2\big[\cos(\theta)^2 + \sin(\theta)^2\big] \\
&= a^2 - 2ab\cos(\theta) + b^2 \\
&= a^2 + b^2 - 2ab\cos(\theta)
\end{aligned}$$

Exercises

Find $\cos(\theta)$.

1.)



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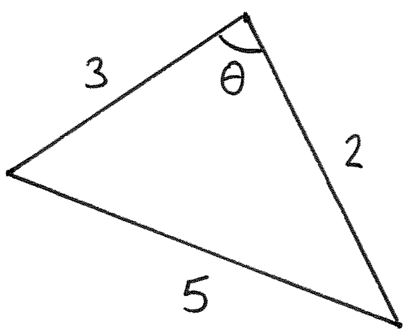

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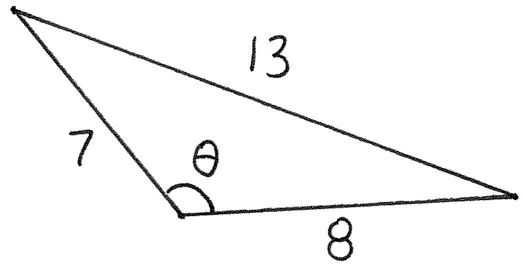

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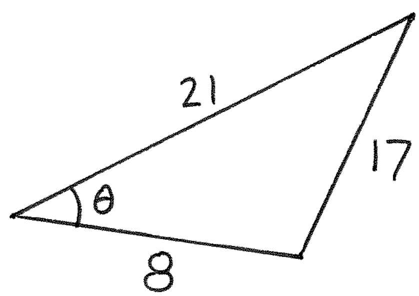
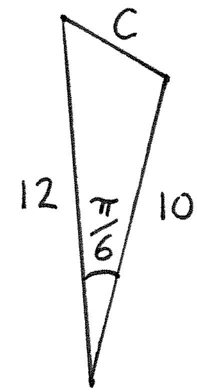


Find c. You can consult the chart on page 227 in the chapter "Sine and Cosine" to find the values of cosine that you need to complete these problems.

5.)



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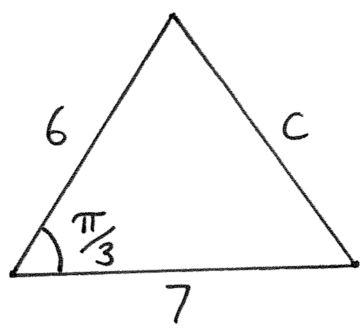

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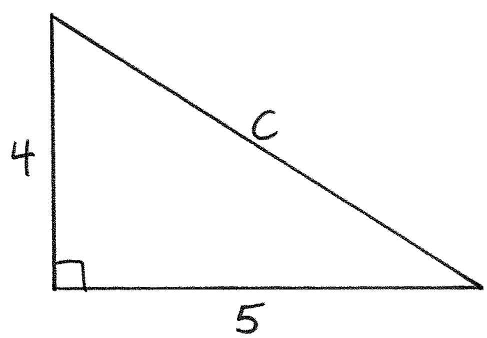

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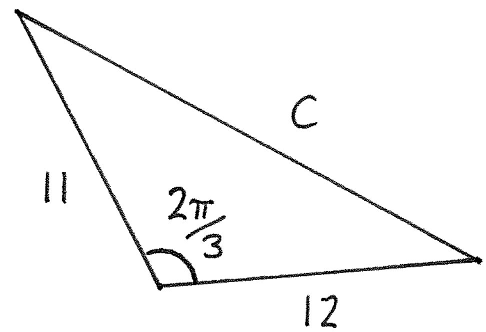

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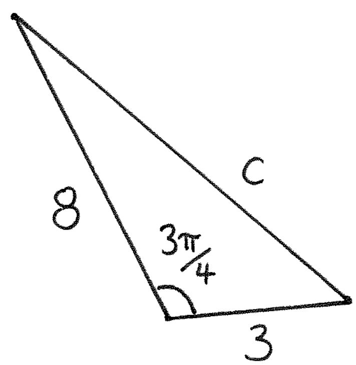

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