Cosecant, Secant, and Cotangent

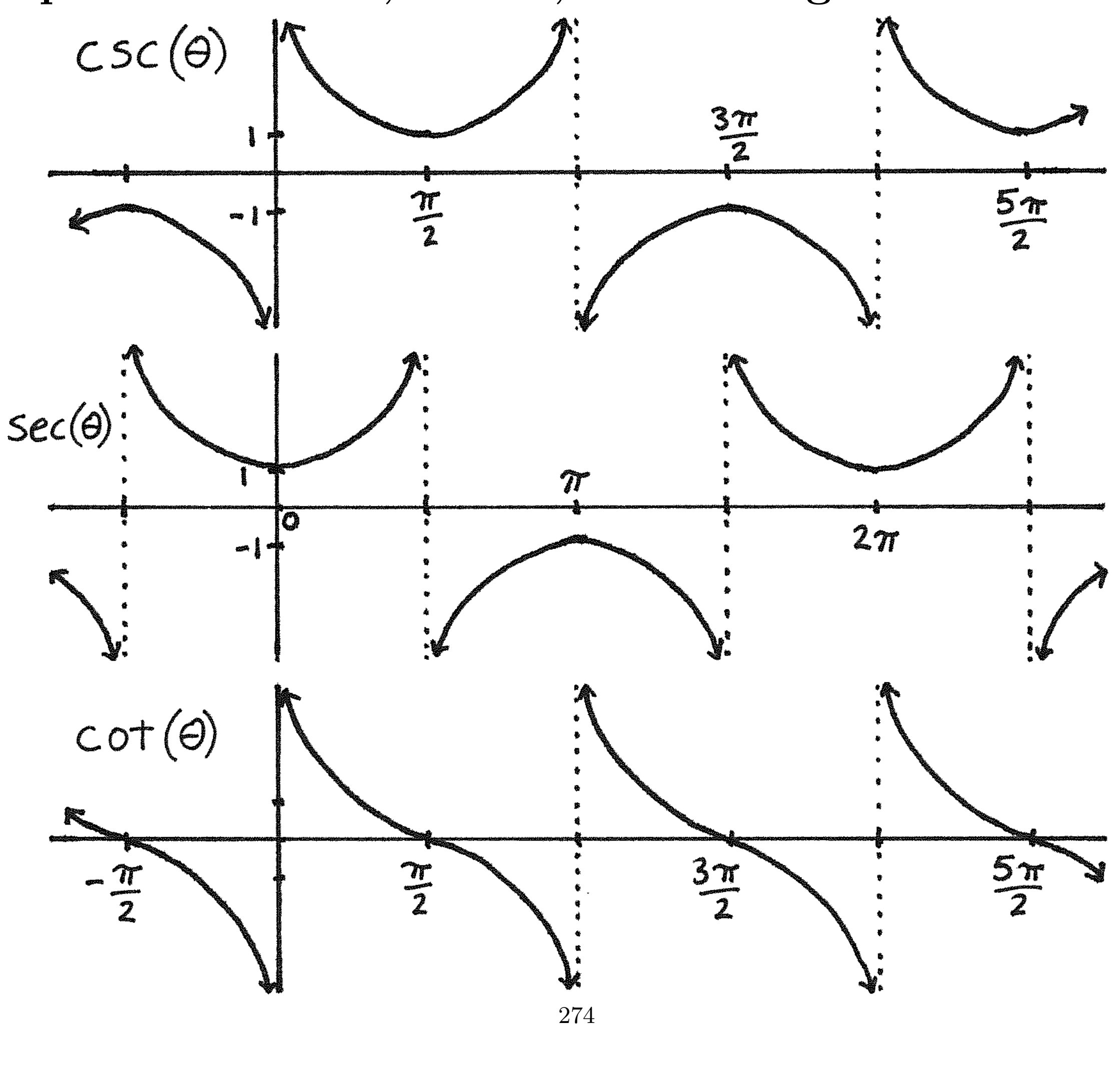
In this chapter we'll introduced three more trigonometric functions: the *cosecant*, the *secant*, and the *cotangent*. These functions are written as $\csc(\theta)$, $\sec(\theta)$, and $\cot(\theta)$ respectively. They are the functions defined by the formulas below:

$$\csc(\theta) = \frac{1}{\sin(\theta)}$$

$$\sec(\theta) = \frac{1}{\cos(\theta)}$$

$$\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$$

Graphs of cosecant, secant, and cotangent



Periods

Cosecant, secant, and cotangent are periodic functions. Cosecant and secant have the same period as sine and cosine do, namely 2π. Cotangent has period π, just as tangent does. In terms of formulas, the previous two sentences mean that

$$\csc(\theta + 2\pi) = \csc(\theta) \qquad \sec(\theta + 2\pi) = \sec(\theta) \qquad \cot(\theta + \pi) = \cot(\theta)$$

It's easy to check why these functions have the periods that they do. For example, because sine has period 2π—that is, because $\sin(\theta + 2\pi) = \sin(\theta)$—we can check that

$$\csc(\theta + 2\pi) = \frac{1}{\sin(\theta + 2\pi)} = \frac{1}{\sin(\theta)} = \csc(\theta)$$

Similarly, the secant function has the same period, 2π, as the function used to define it, cosine.

Even and odd

Recall that an even function is a function $f(x)$ with the property that $f(-x) = f(x)$. Examples include x^2, x^4, x^6, and cosine.

We can add secant to the list of functions that we know are even functions. That is, $\sec(-\theta) = \sec(\theta)$. The reason secant is even is that cosine is even:

$$\sec(-\theta) = \frac{1}{\cos(-\theta)} = \frac{1}{\cos(\theta)} = \sec(\theta)$$

An odd function is a function $f(x)$ with the property that $f(-x) = -f(x)$. Examples include x^3, x^5, x^7, sine, and tangent.

Cosecant and cotangent are odd functions, meaning that $\csc(-\theta) = -\csc(\theta)$ and $\cot(-\theta) = -\cot(\theta)$. We can check that these identities are true by using that sine is an odd function and that cosine is even:

$$\csc(-\theta) = \frac{1}{\sin(-\theta)} = \frac{1}{-\sin(\theta)} = -\csc(\theta)$$

$$\cot(-\theta) = \frac{\cos(-\theta)}{\sin(-\theta)} = \frac{\cos(\theta)}{-\sin(\theta)} = -\cot(\theta)$$

Cofunction identities

Sine and cosine, secant and cosecant, tangent and cotangent; these pairs of functions satisfy a common identity that is sometimes called the *cofunction identity*:

$$\sin\left(\tfrac{\pi}{2}-\theta\right) = \cos(\theta)$$

$$\sec\left(\tfrac{\pi}{2}-\theta\right) = \csc(\theta)$$

$$\tan\left(\tfrac{\pi}{2}-\theta\right) = \cot(\theta)$$

These identities also "go the other way":

$$\cos\left(\tfrac{\pi}{2}-\theta\right) = \sin(\theta)$$

$$\csc\left(\tfrac{\pi}{2}-\theta\right) = \sec(\theta)$$

$$\cot\left(\tfrac{\pi}{2}-\theta\right) = \tan(\theta)$$

Let's check one of these six identities, the identity $\cos\left(\frac{\pi}{2}-\theta\right) = \sin(\theta)$. In order to see that this identity is true, we'll start with $\cos\left(\frac{\pi}{2}-\theta\right)$ and we'll use that cosine is an even function, so

$$\cos\left(\frac{\pi}{2}-\theta\right) = \cos\left(-\left[\frac{\pi}{2}-\theta\right]\right) = \cos\left(\theta-\frac{\pi}{2}\right)$$

Now we can use the identity $\cos\left(\theta-\frac{\pi}{2}\right) = \sin(\theta)$ (which is Lemma 9 from the Sine and Cosine chapter) so that we have

$$\cos\left(\frac{\pi}{2}-\theta\right) = \cos\left(\theta-\frac{\pi}{2}\right) = \sin(\theta)$$

as we had claimed.

Using the cofunction identity that we just examined, $\sin(\theta) = \cos\left(\frac{\pi}{2}-\theta\right)$, we can check that the first cofunction identity from the list above is true:

$$\sin\left(\frac{\pi}{2}-\theta\right) = \cos\left(\frac{\pi}{2}-\left[\frac{\pi}{2}-\theta\right]\right) = \cos(\theta)$$

Exercises

For #1-12, use the chart on page 227 in the chapter "Sine and Cosine" and that $\csc(\theta) = \frac{1}{\sin(\theta)}$, $\sec(\theta) = \frac{1}{\cos(\theta)}$, and $\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$ to find the given value.

1.) $\csc\left(\frac{\pi}{6}\right)$

2.) $\csc\left(\frac{\pi}{4}\right)$

3.) $\csc\left(\frac{\pi}{3}\right)$

4.) $\csc\left(\frac{\pi}{2}\right)$

5.) $\sec(0)$

6.) $\sec\left(\frac{\pi}{6}\right)$

7.) $\sec\left(\frac{\pi}{4}\right)$

8.) $\sec\left(\frac{\pi}{3}\right)$

9.) $\cot\left(\frac{\pi}{6}\right)$

10.) $\cot\left(\frac{\pi}{4}\right)$

11.) $\cot\left(\frac{\pi}{3}\right)$

12.) $\cot\left(\frac{\pi}{2}\right)$

Find the solutions of the following equations in one variable.

13.) $\log_e(x) = \log_e(12) - \log_e(x+1)$

14.) $(x-4)^2 = 36$

15.) $e^{3x-2} = -4$

Match the numbered piecewise defined functions with their lettered graphs below.

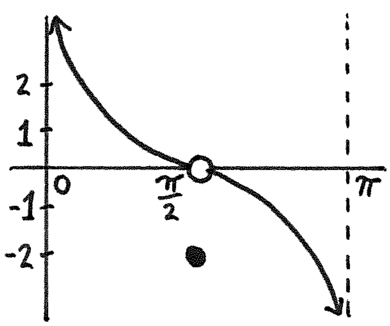
16.) $f(x) = \begin{cases} \csc(x) & \text{if } 0 < x < \frac{\pi}{2} \text{ or } \frac{\pi}{2} < x < \pi \\ 2 & \text{if } x = \frac{\pi}{2} \end{cases}$

17.) $g(x) = \begin{cases} \csc(x) & \text{if } 0 < x < \frac{\pi}{2} \text{ or } \frac{\pi}{2} < x < \pi \\ -2 & \text{if } x = \frac{\pi}{2} \end{cases}$

18.) $h(x) = \begin{cases} \cot(x) & \text{if } 0 < x < \frac{\pi}{2} \text{ or } \frac{\pi}{2} < x < \pi \\ 2 & \text{if } x = \frac{\pi}{2} \end{cases}$

19.) $p(x) = \begin{cases} \cot(x) & \text{if } 0 < x < \frac{\pi}{2} \text{ or } \frac{\pi}{2} < x < \pi \\ -2 & \text{if } x = \frac{\pi}{2} \end{cases}$

A.)



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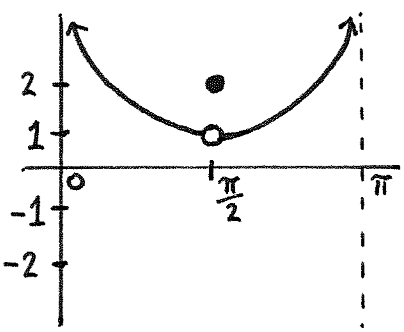
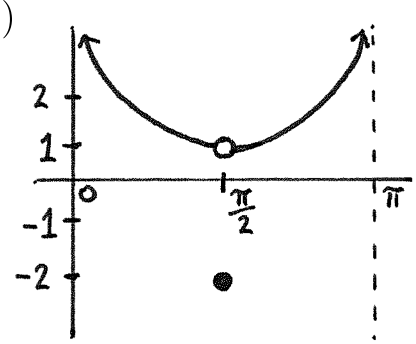


C.)



D.)