Cubic Formula

The complex numbers were created (or they were discovered, depending on your point of view) in the 16th century. Many people will tell you that they were created to provide us with solutions to quadratic equations such as $x^2 = -1$. (There are no real numbers that are solutions to this equation, but $i, -i \in \mathbb{C}$ are both solutions.) While many people will tell you this, it's not exactly true.

The complex numbers were created because they were useful in telling mathematicians about real numbers, which were the numbers they cared about at the time. Specifically, what complex numbers were originally used for was to find real number solutions of cubic equations.

Mathematicians knew how to solve quadratic equations, equations of the form $ax^2 + bx + c = 0$. They knew the quadratic formula, and the quadratic formula tells you anything you want to know about the solutions of quadratic equations. What they didn't know was a cubic formula, a formula that could solve cubic equations like $ax^3 + bx^2 + cx + d = 0$. (Here a, b, c, and d are real numbers.) Mathematicians knew that cubic equations had solutions, and they knew that the solutions of the equations they were interested in were real numbers, but they couldn't figure out what the solutions were until they created the cubic formula. They discovered the formula in the 16th century, and along with it the complex numbers were discovered, because it's the arithmetic of the complex numbers that makes the cubic formula work, even if you're only interested in finding solutions that are real numbers.

In this chapter, we'll see how the cubic formula works, and how it uses complex numbers to tell us about real numbers.

A Computation we'll need later

Before we begin, we should make a quick digression. We'll want to find $(2+i)^3$. We have two options. We can multiply it out as $(2+i)(2+i)(2+i)$ using the distributive law many times. Or, we can use the binomial theorem. You may have seen the binomial theorem in Math 1050. It says that

$$(2+i)^3 = 2^3 + 3(2^2)i + 3(2)i^2 + i^3$$

Simplifying, we have

$$\begin{aligned}(2+i)^3 &= 8+12i+6i^2+i^3 \\ &= 8+i12+i^2 6+i^2 i \\ &= 8+i12-6-i \\ &= 2+i11\end{aligned}$$

We'll need to know that $(2+i)^3 = 2+i11$ later in this chapter.

A Simplification for cubics

In the chapter on Classification of Conics, we saw that any quadratic equation in two variables can be modified to one of a few easy equations to understand. In a similar process, mathematicians had known that any cubic equation in one variable—an equation of the form $ax^3+bx^2+cx+d=0$—could be modified to look like a cubic equation of the form $x^3+ax+b=0$, so it was these simpler cubic equations that they were looking for solutions of. If they could find the solutions of equations of the form $x^3+ax+b=0$ then they would be able to find the solutions of any cubic equation.

Of the simpler cubic equations that they were trying to solve, there was an easier sort of equation to solve, and a more complicated sort. The easier sort were equations of the form $x^3+ax+b=0$ where $-\left(\frac{a}{3}\right)^3-\left(\frac{b}{2}\right)^2 \leq 0$. The more complicated sort were equations $x^3+ax+b=0$ where $-\left(\frac{a}{3}\right)^3-\left(\frac{b}{2}\right)^2$ was a positive number. We're going to focus on the more complicated sort when discussing the cubic formula. Figuring out how to solve these more complicated equations was the key to figuring out how to solve any cubic equation, and it was the cubic formula for these equations that lead to the discovery of complex numbers.

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The Cubic formula

Here are the steps for finding the solutions of a cubic equation of the form

$$x^3 + ax + b = 0 \quad \text{if} \quad -\left(\frac{a}{3}\right)^3 - \left(\frac{b}{2}\right)^2 > 0$$

Step 1. Let D be the complex number

$$D = -\frac{b}{2} + i\sqrt{-\left(\frac{a}{3}\right)^3 - \left(\frac{b}{2}\right)^2}$$

Step 2. Find a complex number $z \in \mathbb{C}$ such that $z^3 = D$.

Step 3. Let R be the real part of z, and let I be the imaginary part of z, so that R and I are real numbers with $z = R + iI$.

Step 4. The three solutions of $x^3 + ax + b = 0$ are the real numbers $2R$, $-R + \sqrt{3}I$, and $-R - \sqrt{3}I$.

These four steps together are the *cubic formula*. It uses complex numbers (D and z) to create real numbers ($2R$, $-R + \sqrt{3}I$, and $-R - \sqrt{3}I$) that are solutions of the cubic equation $x^3 + ax + b = 0$.

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We'll take a look at two examples of cubic equations, and we'll use the cubic formula to find their solutions.

First example

In this example we'll use the cubic formula to find the solutions of the equation

$$x^3 - 15x - 4 = 0$$

Notice that this is a cubic equation $x^3 + ax + b = 0$ where $a = -15$ and $b = -4$. Thus,

$$\begin{aligned} -\left(\frac{a}{3}\right)^3 - \left(\frac{b}{2}\right)^2 &= -\left(\frac{-15}{3}\right)^3 - \left(\frac{-4}{2}\right)^2 \\ &= -(-5)^3 - (-2)^2 \\ &= -(-125) - (4) \\ &= 125 - 4 \\ &= 121 \end{aligned}$$

and 121 is a positive number, so the cubic equation $x^3 - 15x - 4 = 0$ is one of the more complicated sorts of equations to solve, and it's equations like these that the cubic formula on the previous page is designed to solve. Now let's go through the four steps of the cubic equation and find the solutions of $x^3 - 15x - 4 = 0$.

Step 1. We need to find the complex number D. It's given by

$$\begin{aligned} D &= -\frac{b}{2} + i\sqrt{-\left(\frac{a}{3}\right)^3 - \left(\frac{b}{2}\right)^2} \\ &= -\frac{-4}{2} + i\sqrt{121} \\ &= 2 + i11 \end{aligned}$$

Step 2. We need to find a complex number z such that $z^3 = 2 + i11$. We saw earlier in this chapter that $(2+i)^3 = 2 + i11$, so we can choose

$$z = 2 + i$$

Step 3. In this step, we write down the real (R) and imaginary (I) parts of $z = 2 + i$. The real part is 2, and the imaginary part is 1. So

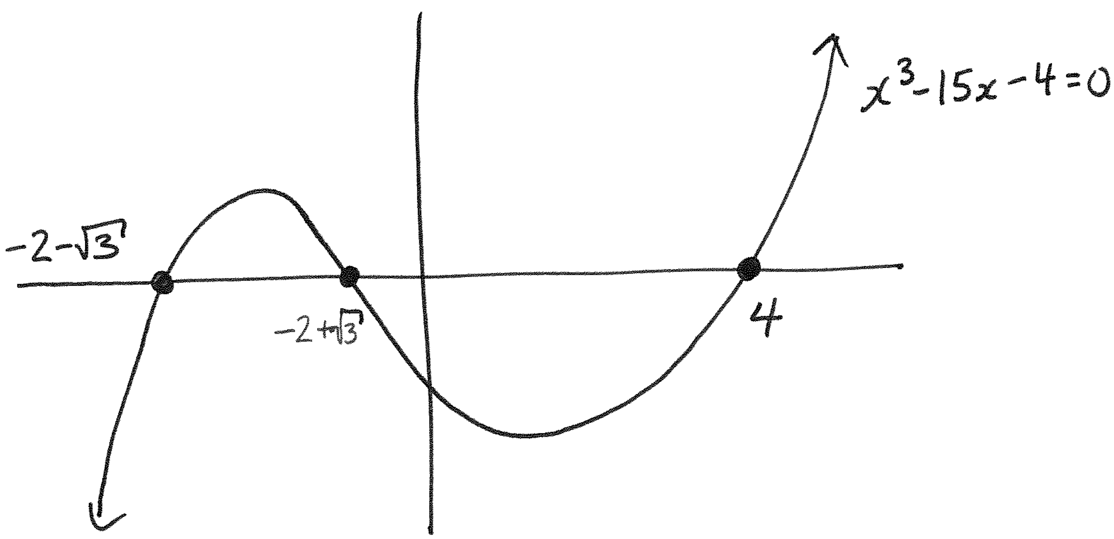
$$R = 2 \quad \text{and} \quad I = 1$$

Step 4. The three solutions of the equation $x^3 - 15x - 4 = 0$ are the real numbers

$$2R = 2(2) = 4$$

$$-R + \sqrt{3}I = -(2) + \sqrt{3}(1) = -2 + \sqrt{3}$$

$$-R + \sqrt{3}I = -(2) - \sqrt{3}(1) = -2 - \sqrt{3}$$



We found the real number solutions— 4, $-2 + \sqrt{3}$, and $-2 - \sqrt{3}$ —of an equation that had real number coefficients— -15 and -4 —using complex numbers— $2 + i11$ and $2 + 1$. This is why the complex numbers seemed attractive to mathematicians originally. They cared about complex numbers because they cared about real numbers, and complex numbers were a tool designed to give them information about real numbers.

Over the last several centuries, complex numbers have proved their usefulness in many other ways in mathematics. They are now viewed as being just as important as the real numbers are.

Second example

The previous example used a shortcut. In Step 2 of that example we didn't really have to go through any work to find the number z that had the property that $z^3 = D$. It was just given to us. In this second example, we'll find the number z in Step 2 from scratch.

Our second example of a cubic equation is $x^3 - 3x + \sqrt{2} = 0$.

It's an equation of the form $x^3 + ax + b$ where $a = -3$ and $b = \sqrt{2}$. To see if we can use the cubic formula given in this chapter, we need to see if the following number is positive:

$$\begin{aligned}
-\Big(\frac{a}{3}\Big)^3 - \Big(\frac{b}{2}\Big)^2 &= -\Big(\frac{-3}{3}\Big)^3 - \Big(\frac{\sqrt{2}}{2}\Big)^2 \\
&= -(-1)^3 - \frac{(\sqrt{2})^2}{2^2} \\
&= -(-1) - \frac{2}{4} \\
&= 1 - \frac{1}{2} \\
&= \frac{1}{2}
\end{aligned}$$

Of course $\frac{1}{2}$ is positive, so we can proceed with the four steps of the cubic formula to find the solutions of $x^3 - 3x + \sqrt{2} = 0$.

Step 1. We need to find the number D. Using the equation from Step 1,

$$\begin{aligned}
D &= -\frac{b}{2} + i\sqrt{-\Big(\frac{a}{3}\Big)^3 - \Big(\frac{b}{2}\Big)^2} \\
&= -\frac{\sqrt{2}}{2} + i\sqrt{\frac{1}{2}} \\
&= -\frac{\sqrt{2}}{\sqrt{2}\sqrt{2}} + i\frac{\sqrt{1}}{\sqrt{2}} \\
&= -\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}
\end{aligned}$$

Step 2. We need to find a complex number z such that $z^3 = -\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$. This will take a little bit of work.

Any complex number can be written in polar coordinates, so let's write z in polar coordinates:

$$z = r(\cos(\theta) + i\sin(\theta))$$

Notice that $-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} = \cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)$ so to say that we want $z^3 = -\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$, or equivalently that $-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} = z^3$, is to say that we want

$$\begin{aligned}\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right) &= [r(\cos(\theta) + i\sin(\theta))]^3 \\ &= r^3(\cos(\theta) + i\sin(\theta))^3\end{aligned}$$

Using De Moivre's Formula, that means

$$\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right) = r^3(\cos(3\theta) + i\sin(3\theta))$$

The complex number on the left of the equation above has norm 1, so the complex number on the right that it equals must also have norm 1. That is, $r^3 = 1$, which implies that $r = 1$. Therefore,

$$\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right) = \cos(3\theta) + i\sin(3\theta)$$

The above equation equates two points on the unit circle. We can do this by letting 3θ equal $\frac{3\pi}{4}$ so that

$$\theta = \frac{1}{3}\left(\frac{3\pi}{4}\right) = \frac{\pi}{4}$$

Now we have found that

$$\begin{aligned}z &= r(\cos(\theta) + i\sin(\theta)) \\ &= 1\Big(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\Big) \\ &= \cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right) \\ &= \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\end{aligned}$$

Step 3. The real part of $z = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$ is $\frac{1}{\sqrt{2}}$, and it's imaginary part also equals $\frac{1}{\sqrt{2}}$. That is,

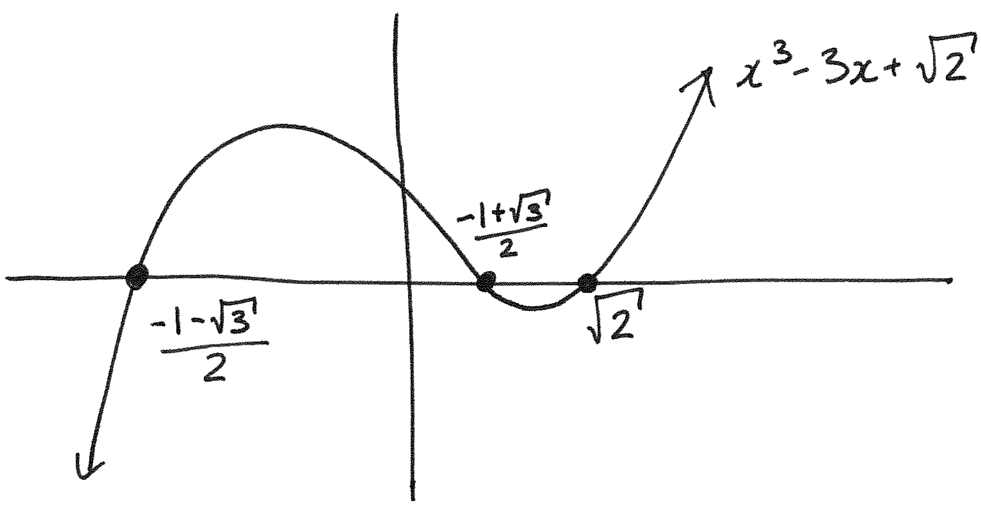
$$R = I = \frac{1}{\sqrt{2}}$$

Step 4. The three real number solutions of $x^3 - 3x + \sqrt{2} = 0$ are

$$2R = 2\Big(\frac{1}{\sqrt{2}}\Big) = \frac{2}{\sqrt{2}} = \frac{\sqrt{2}\sqrt{2}}{\sqrt{2}} = \sqrt{2}$$

$$-R + \sqrt{3}I = -\Big(\frac{1}{\sqrt{2}}\Big) + \sqrt{3}\Big(\frac{1}{\sqrt{2}}\Big) = \frac{-1+\sqrt{3}}{\sqrt{2}}$$

$$-R - \sqrt{3}I = -\Big(\frac{1}{\sqrt{2}}\Big) - \sqrt{3}\Big(\frac{1}{\sqrt{2}}\Big) = \frac{-1-\sqrt{3}}{\sqrt{2}}$$



As this example illustrates, the major work in applying the cubic formula comes in Step 2, solving the equation of complex numbers $z^3 = D$.

Exercises

Use the First example from this chapter to find the solutions of the following equations.

1.) $(x-2)^3 - 15(x-2) - 4 = 0$

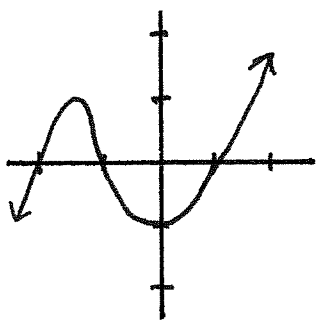
2.) $\log_e(x)^3 - 15\log_e(x) - 4 = 0$

All further exercises in this chapter have nothing to do with complex numbers.

Match the functions with their graphs.

3.) $f(x)+1$	4.) $f(2x)$	5.) $f(x+1)$
6.) $\frac{1}{2}f(x)$	7.) $f(x)-1$	8.) $2f(x)$
9.) $f(x-1)$	10.) $f\left(\frac{x}{2}\right)$	

$f(x)$



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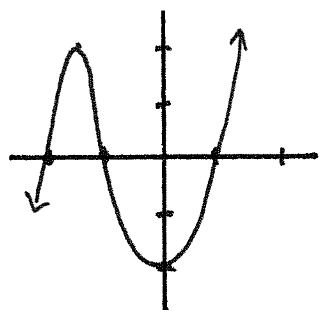

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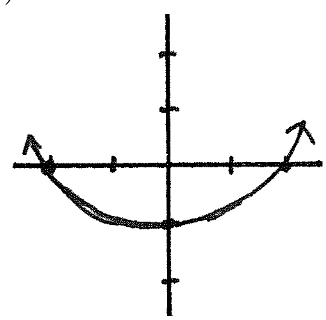
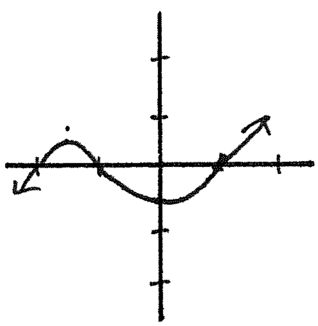
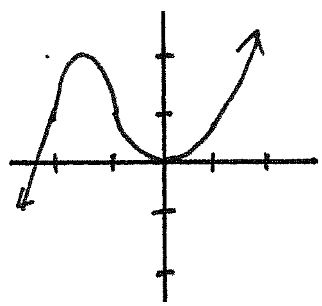


C.)



D.)



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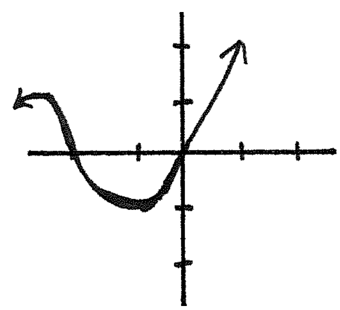
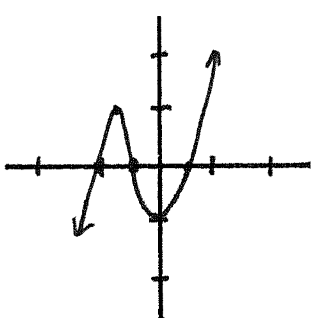
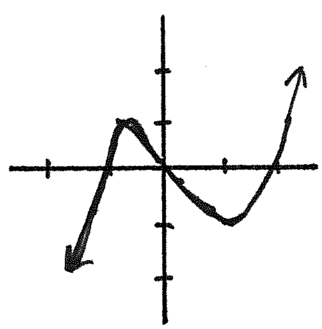


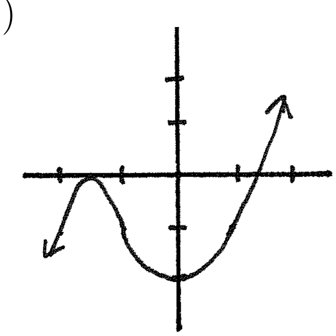
F.)



G.)



H.)



Match the functions with their graphs.

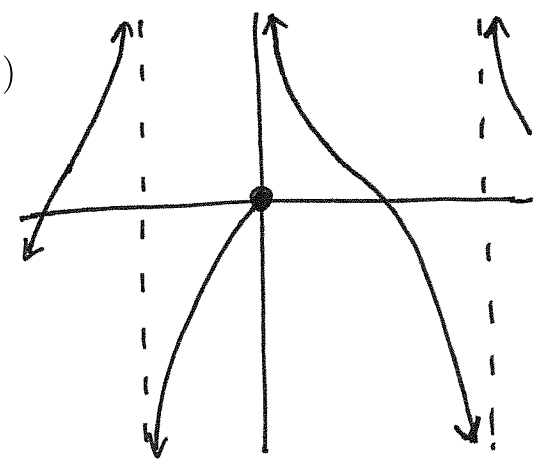
11.) $\tan(x)$

12.) $\cot(x)$

13.) $f(x) = \begin{cases} \tan(x) & \text{if } x \leq 0; \\ \cot(x) & \text{if } x > 0. \end{cases}$

14.) $g(x) = \begin{cases} \cot(x) & \text{if } x < 0; \\ \tan(x) & \text{if } x \geq 0. \end{cases}$

A.)



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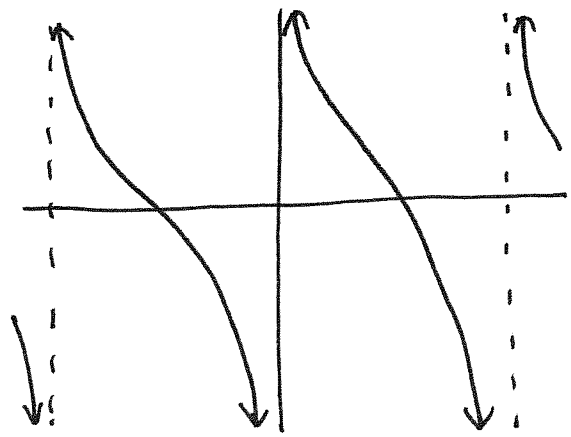

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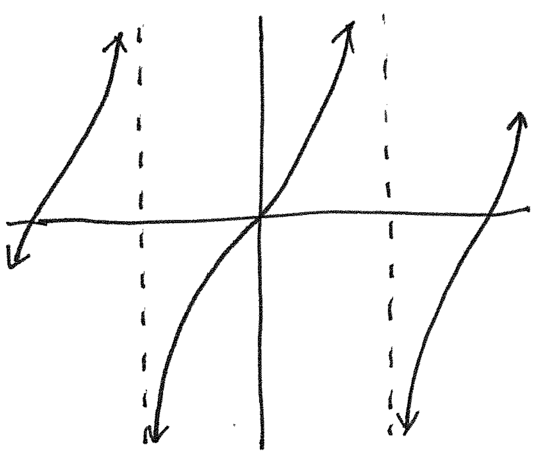
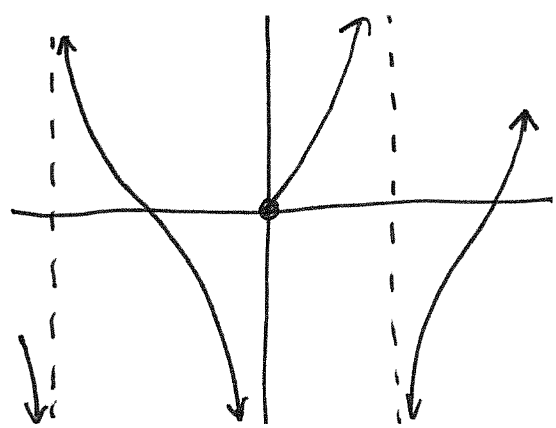


D.)



To find the solutions of an equation of the form $h(x)f(x) = h(x)g(x)$, you need to find the solutions of the following two equations:

$$h(x) = 0 \quad \text{and} \quad f(x) = g(x)$$

Find the solutions of the following equations.

15.) $(x-1)(x+3) = (x-1)(x+4)$

16.) $(2x-3)(x+1) = (2x-3)(x+2)$

17.) $(x-7) = (x-7)(x+2)$

18.) $x^2 = x$

In the exercises from the last chapter we reviewed rules for when some equations have solutions that can be found in one step. Those rules can be expanded on to give us rules for a single step of a problem that might involve several steps. These rules are listed below. Use them to solve the equations in the remaining exercises.

- $f(x)^2 = c$ implies $f(x) = \sqrt{c}$ or $f(x) = -\sqrt{c}$
- $af(x)^2 + bf(x) + c = 0$ implies $f(x) = \frac{-b-\sqrt{b^2-4ac}}{2a}$ or $f(x) = \frac{-b+\sqrt{b^2-4ac}}{2a}$
- $\sqrt{f(x)} = c$ implies $f(x) = c^2$

19.) $(2x-3)^2 = 4$

20.) $\sqrt{x+4} = 3$

21.) $(e^x)^2 - 3e^x + 2 = 0$

22.) $(\sqrt{x})^2 - 5\sqrt{x} + 6 = 0$

23.) $\log_e(x)^2 = 25$

24.) $\sqrt{2x-3} = 5$