UNIFORM HYPERBOLICITY, COCYCLES AND RIGIDITY - PRELIMINARIES ON CODING

Definitions and Main Theorems

Theorem 0.1 (Hadamard-Perron Theorem). *If $0 < \lambda < 1$ and $\gamma > 0$, then there exists $\eta = \eta(\lambda, \gamma)$ such that the following holds: Let $F_n : \mathbb{R}^m \times \mathbb{R}^{d-m} \to \mathbb{R}^m \times \mathbb{R}^{d-m}$, $n \in \mathbb{Z}$, be a sequence of C^r maps satsifying:*

$$F_n(u,v) = (u,v) \cdot \begin{pmatrix} A_n & 0 \\ 0 & B_n \end{pmatrix} + \delta_n(u,v)$$

where $A_n \in GL(m,\mathbb{R})$ and $B_n \in GL(d-m,\mathbb{R})$ satisfy $\left|\left|{A_n}^{-1}\right|\right| \leq \lambda$ and $||B_n|| \leq \lambda$ for every n, and $\delta_n : \mathbb{R}^d \to \mathbb{R}^m \times \mathbb{R}^d$ satsifes $||\delta_n||_{C^1} < \eta$ for every n. Then there exist a sequence of C^r functions $\varphi_n^u : \mathbb{R}^m \to \mathbb{R}^{d-m}$ and $\varphi_n^s : \mathbb{R}^{d-m} \to \mathbb{R}^m$ such that $||d\varphi_n^*|| \leq \gamma$ for every n, and if $W_n^u = \mathrm{graph}(\varphi_n^u) = \{(x, \varphi_n^u(x))\} \subset \mathbb{R}^d$ and $W_n^s = \mathrm{graph}(\varphi_n^s) = \{(\varphi_n^s(y), y)\} \subset \mathbb{R}^d$, then $F_n(W_n^*) = W_n^*$, and

$$\begin{aligned} W_n^s &= \left\{x \in \mathbb{R}^d : \lim_{m\to\infty} F_{n+m} \circ \ldots \circ F_n(x) \to 0\right\} \\ W_n^u &= \left\{x \in \mathbb{R}^d : \lim_{m\to\infty} F_{n-m} \circ \ldots \circ F_{n-1}^{-1}(x) \to 0\right\} \end{aligned}$$

Corollary 0.2 (Stable and Unstable Manifold Theorem). *If $f : X \to X$ is a C^r Anosov diffeomorphism, there exists some $\varepsilon > 0$ such that for every $x \in X$ there exist C^r-embedded manifolds $W^u_{\mathrm{loc}}(x)$ and $W^s_{\mathrm{loc}}(x)$ such that $T_y W^*_{\mathrm{loc}}(x) = E^*(y)$ for every $y \in W^*_{\mathrm{loc}}(x)$, $* = s, u$ and if $d(y,x) < \varepsilon$, then $d(f^n(x), f^n(y)) < \varepsilon$ for every $n \geq 0$ and converges to 0 if and only if $y \in W^s_{\mathrm{loc}}(x)$ and $d(f^{-n}(x), f^{-n}(y)) \leq \varepsilon$ for every $n \geq 0$ and converges to 0 if and only if $y \in W^u_{\mathrm{loc}}(x)$. They are unique in sufficiently small neighborhoods of x.*

Definition 0.3. Fix an alphabet $\mathcal{A} = \{1, \ldots, m\}$. Let $\Sigma_+ = \{(x_n)_{n=0}^\infty : x_n \in \mathcal{A} \text{ for every } n \in \mathbb{N}_0\}$ and $\Sigma = \left\{(x_n)_{n=-\infty}^\infty : x_n \in \mathcal{A} \text{ for every } n \in \mathbb{Z}\right\}$ be the set of *one-sided sequences* and *two-sided sequences*, respectively. Each comes equipped with the *left-shift map*

$$\sigma((x_n))_\ell = x_{\ell+1}.$$

Given an $m \times m$ matrix A, whose entries are all 0 or 1, call a (finite or infinite) sequence (x_n) *A-admissible* if $A_{x_n x_{n+1}} = 1$ for every relevant n. The *Markov shift* or *subshift of finite type* determined by A is the subset of Σ (or Σ_+) of A-admissible sequences, denoted by Σ^A (or Σ_+^A, respectively).

Definition 0.4. If $f : X \to X$ is a transformation of a compact metric space X, define a family of metric $d_{f,n}$ by $d_{f,n}(x,y) = \inf \{d(f^i(x), f^i(y)) : i = 0, \dots, n-1\}$, and a *Bowen ball* at x of depth n to be the set $B_{f,n}(x,\varepsilon) = \{y \in X : d(f^i x, f^i y) < \varepsilon\}$ for every $i = 0, \dots, n-1$.

An (n,ε)-net is a finite collection of points $\{x_j\}$ such that $X = \bigcup B_{f,n}(x_j)$. An (n,ε)-separated set is a finite collection of points $\{x_j\}$ such that the sets $B_{f,n}(x_j)$ are pairwise disjoint. Let $N(f,n,\varepsilon)$ denote minimal cardinality of an (n,ε)-net and $S(f,n,\varepsilon)$ denote the maximal cardinality of an (n,ε) separated set.

Define the *topological entropy of* f as:

$$h_{\text{top}}(f) := \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log N(f,n,\varepsilon) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log S(f,n,\varepsilon)$$

Theorem 0.5. *The topological entropy of a topologically mixing subshift of finite type* Σ^A *is the largest eigenvlue of* A.

Definition 0.6. Given a finite partition of a measure space (X,μ) into measurable subsets $\mathcal{P} = \{P_1, \dots, P_n\}$, let $H(\mathcal{P}) = \sum I(\mu(P))$, where $I(x) = -x \log x$. The *wedge* of partitions $\mathcal{Q}_1, \dots, \mathcal{Q}_m$ is denoted by $\mathcal{Q}_1 \vee \cdots \vee \mathcal{Q}_m$, and the atoms of the partition are sets of the form $Q_1 \cap \cdots \cap Q_m$, where $Q_i \in \mathcal{Q}_i$ for every i. Given a μ-preserving transformation $T : X \to X$, let $\mathcal{P}_n^T$ denote the partition $\mathcal{P}_n^T = \mathcal{P} \vee T^{-1}(\mathcal{P}) \vee \cdots \vee T^{-(n-1)}(\mathcal{P})$.

The *metric entropy* or *measure-theoretic entropy* of T with respect to μ is:

$$h_\mu(f) := \sup_{\mathcal{P}} \lim_{n\to\infty} \frac{1}{n} H(\mathcal{P}_n^T).$$

A matrix B is called a *stochastic matrix (subordinate to A)* if all of its entries are nonnegative, the rows of B sum to 1, and $B_{ij} = 0$ if and only if $A_{ij} = 0$.

Theorem 0.7. *Given a stochastic matrix* B *subordinate to* A*, if* Σ^A *is topologically mixing, there exists a unique left eigenvector* p *of* B*, and a* σ*-invariant probability measure* μ_B *on* Σ^A *such that for every finite* A*-admissible word* $\omega = (\omega_1, \dots, \omega_n)$*, if* $C_\omega = \{(x_n) \in \Sigma^A : (x_n)$ *begins with* $\omega\}$*, then*

$$\mu_B(C_\omega) = p_{\omega_1} \cdot \prod_{i=1}^{n-1} B_{\omega_i \omega_{i+1}}.$$

The metric entropy of σ *with respect to* μ_B *is* $\sum_{i,j} p_i I(B_{ij})$.

Exercises

Problem 1. Consider the function $d((x_n),(y_n)) = \begin{cases} 0, & x_n = y_n \text{ for every } n \\ 2^{-\ell}, & \text{where } \ell = \inf\{|n| : x_n \neq y_n\} \end{cases}$ Prove that d is a metric on Σ_+ and Σ. Describe $B((x_n),\varepsilon)$ for an arbitrary sequence (x_n). Use this description to show that d induces the product topology, when Σ_+ and Σ are viewed as $\mathcal{A}^{\mathbb{N}_0}$ and $\mathcal{A}^{\mathbb{Z}}$, respectively. Show that the subshifts $\Sigma^A \subset \Sigma$ and $\Sigma_+^A \subset \Sigma_+$ are closed, σ-invariant subsets.

Problem 2. Assume that a 0-1 matrix A has no rows or columns which have all zeroes. Show that a subshift Σ^A is topologically transitive if and only if for every pair $1 \leq i, j \leq m$ there exists some $n \in \mathbb{N}$ such that $(A^n)_{ij} \neq 0$. Show that it is topologically mixing if and only if there exists some $n \in \mathbb{N}$ such that every entry of A^n is nonzero.

Problem 3. Show that if f is an Anosov diffeomorphism of a compact manifold, then for sufficiently small ε, $\bigcap_{n\in\mathbb{N}} B_{f,n}(x,\varepsilon) = W^s_{\text{loc}}(x)$.

Problem 4. Let $f : X \to X$ and $g : Y \to Y$ be transformations of compact metric spaces. g is called a *factor* of f if there exists a continuous surjection $\pi : X \to Y$ such that $\pi \circ f = g \circ \pi$. Show that if g is a factor of f, then $h_{\text{top}}(g) \leq h_{\text{top}}(f)$. Furthermore, show that if there exists B such that $\left|\pi^{-1}(x)\right| \leq B$, then $h_{\text{top}}(g) = h_{\text{top}}(f)$.

Problem 5. Show that if $\mathcal{A} = \{0,1\}$, then the linear expanding map on the circle $L_2(x) = 2x \pmod 1$ is a factor of Σ with factor map $\pi((x_n)) = \sum_{n=0}^{\infty} 2^{-(n+1)} x_n$. Furthermore, show that π is one-to-one, with the exception of a countable subset, on which π is 2-to-1.

Problem 6. Show that if $f : X \to X$ is a homeomorphism of a compact metric space, then $h_{\text{top}}(f) = h_{\text{top}}(f^{-1})$.

Problem 7 (Baby Brin-Katok Theorem/Pesin Entropy Formula)**.** * Assume that X is a compact Riemannian manifold, and $f : X \to X$ is an Anosov diffeomorphism such that for every $x \in X$, if $v \in E^u_x$ then $||df(v)|| = \lambda\,||v||$ and if $w \in E^s_x$, then $||df(w)|| = \mu\,||w||$ for some fixed $\lambda > 1$ and $\mu < 1$ indpendent of x. Furthermore, assume that f preserves the Riemannian volume (ie, $f^*\,\text{vol} = \text{vol}$). Show that there exist $0 < c < C$ such that for every $x \in X$, $c\lambda^{dn} \leq \text{vol}(B_{f,n}(x,\varepsilon)) \leq C\lambda^{dn}$, where $d = \dim(E^u)$. Conclude that the topological entropy of f is $d \log \lambda$.

Problem 8. * Fix a length N and let $\mathcal{L} \subset \mathcal{A}^N$ be a *language* consisting of words of length N. Say that a word $(x_n) \in \Sigma$ is $\mathcal{L}$-admissible if for every m, $(x_m, \dots, x_{m+n-1})$ is in $\mathcal{L}$. Let $\Sigma^{\mathcal{L}}$ denote the set of infinite, $\mathcal{L}$-admissible sequences, and show that $\mathcal{L}$ is shift-invariant and closed.

Now consider the set $\mathcal{L}$ as the alphabet to define a shift, so that the letters of the alphabet are the collection $\mathcal{L}$ of words of length N in $\mathcal{A}$. Define the adjacency matrix $A(\mathcal{L})$ such that if $\sigma_1, \sigma_2 \in \mathcal{L}$, then $A_{\sigma_1\sigma_2} = 1$ if and only if the last $N-1$ letters of σ_1 and first $N-1$ letters of σ_2 coincide. Show that the shifts on $\Sigma^{A(\mathcal{L})}$ and $\Sigma^{\mathcal{L}}$ are topologically conjugate. **Moral:** All finite language subshifts can be studied using 2-step subshifts.

Problem 9. * Prove that the global stable and unstable manifolds form a foliation using the following scheme:

(1) Show that if $W^s_{\text{loc}}(x_1) \cap W^s_{\text{loc}}(x_2) \neq \emptyset$, then their union is a connected manifold of the same dimension (Use the uniquness and dynamical characterization)
(2) Say that $W^s(x) = \{y \in X : d(f^n(x), f^n(y)) \to 0\}$. Show that if $z \in W^s(x)$, then $W^s_{\text{loc}}(z) \subset W^s(x)$. Define a topology on $W^s(x)$ in which $d(z_1,z_2) < \varepsilon$ if and only if $z_1 \in W^s_{\text{loc}}(z_2)$ and the points are ε-close on the local stable manifold.
(3) Prove that $W^s(x)$ is an immersed submanifold without self-intersections.

(4) Given $x \in X$, define a map $\psi_x : W^s_{\mathrm{loc}}(x) \times W^u_{\mathrm{loc}}(x) \to Y$ by $(y, z) \mapsto W^u_{\mathrm{loc}}(y) \cap W^s_{\mathrm{loc}}(x)$. Prove that the map is well-defined and forms a foliation atlas, so that each W^u and W^s are continuous foliations with C^r-leaves.