UNIFORM HYPERBOLICITY, COCYCLES AND RIGIDITY - PRELIMINARIES ON CODING

Definitions and Main Theorems

**Theorem 0.1** (Hadamard-Perron Theorem). If $0 < \lambda < 1$ and $\gamma > 0$, then there exists $\eta = \eta(\lambda, \gamma)$ such that the following holds: Let $F_n : \mathbb{R}^m \times \mathbb{R}^{d-m} \to \mathbb{R}^m \times \mathbb{R}^{d-m}$, $n \in \mathbb{Z}$, be a sequence of $C^r$ maps satisfying:

$$F_n(u, v) = (u, v) \cdot \begin{pmatrix} A_n & 0 \\ 0 & B_n \end{pmatrix} + \delta_n(u, v)$$

where $A_n \in GL(m, \mathbb{R})$ and $B_n \in GL(d-m, \mathbb{R})$ satisfy $||A_n^{-1}|| \leq \lambda$ and $||B_n|| \leq \lambda$ for every $n$, and $\delta_n : \mathbb{R}^d \to \mathbb{R}^m \times \mathbb{R}^d$ satisfies $||\delta_n||_{C^1} < \eta$ for every $n$. Then there exists a sequence of $C^r$ functions $\varphi_n^s : \mathbb{R}^m \to \mathbb{R}^{d-m}$ and $\varphi_n^u : \mathbb{R}^{d-m} \to \mathbb{R}^m$ such that $||d\varphi_n^s|| \leq \gamma$ for every $n$, and if $W_u = \text{graph}(\varphi_n^u) = \{(x, \varphi_n^u(x))\} \subset \mathbb{R}^d$ and $W_s = \text{graph}(\varphi_n^s) = \{(\varphi_n^s(y), y)\} \subset \mathbb{R}^d$, then $F_n(W_u) = W_u^s$, and

$$W_n^s = \left\{ x \in \mathbb{R}^d : \lim_{m \to \infty} F_{n+m} \circ \ldots \circ F_n(x) \to 0 \right\}$$

$$W_n^u = \left\{ x \in \mathbb{R}^d : \lim_{m \to \infty} F_{n-m} \circ \ldots \circ F_{n-1}(x) \to 0 \right\}$$

**Corollary 0.2** (Stable and Unstable Manifold Theorem). If $f : X \to X$ is a $C^r$ Anosov diffeomorphism, there exists some $\varepsilon > 0$ such that for every $x \in X$ there exist $C^{r}$-embedded manifolds $W_{\text{loc}}^s(x)$ and $W_{\text{loc}}^u(x)$ such that $T_y W_{\text{loc}}^s(x) = E^s(y)$ for every $y \in W_{\text{loc}}^s(x)$, $* = s, u$ and if $d(y, x) < \varepsilon$, then $d(f^m(y), f^m(x)) < \varepsilon$ for every $m \geq 0$ and converges to 0 if and only if $y \in W_{\text{loc}}^s(x)$ and $d(f^{-m}(x), f^{-m}(y)) \leq \varepsilon$ for every $m \geq 0$ and converges to 0 if and only if $y \in W_{\text{loc}}^u(x)$. They are unique in sufficiently small neighborhoods of $x$.

**Definition 0.3.** Fix an alphabet $\mathcal{A} = \{1, \ldots, m\}$. Let $\Sigma_+ = \{(x_n)_{n=0}^\infty : x_n \in \mathcal{A}$ for every $n \in \mathbb{N}_0\}$ and $\Sigma = \{(x_n)_{n=-\infty}^\infty : x_n \in \mathcal{A}$ for every $n \in \mathbb{Z}\}$ be the set of one-sided sequences and two-sided sequences, respectively. Each comes equipped with the left-shift map $\sigma((x_n))_\ell = x_{\ell+1}$.

Given an $m \times m$ matrix $A$, whose entries are all 0 or 1, call a (finite or infinite) sequence $(x_n)$ $A$-admissible if $A_{x_n x_{n+1}} = 1$ for every relevant $n$. The Markov shift or subshift of finite type determined by $A$ is the subset of $\Sigma$ (or $\Sigma_+$) of $A$-admissible sequences, denoted by $\Sigma_A$ (or $\Sigma^A_+$, respectively).
Definition 0.4. If \( f : X \to X \) is a transformation of a compact metric space \( X \), define a family of metric \( d_{f,n} \) by \( d_{f,n}(x,y) = \inf \{ d(f^i(x), f^i(y)) : i = 0, \ldots, n-1 \} \), and a Bowen ball at \( x \) of depth \( n \) to be the set \( B_{f,n}(x, \varepsilon) = \{ y \in X : d(f^i(x), f^i(y)) < \varepsilon \} \) for every \( i = 0, \ldots, n-1 \).

An \((n, \varepsilon)\)-net is a finite collection of points \( \{x_j\} \) such that \( X = \bigcup B_{f,n}(x_j) \). An \((n, \varepsilon)\)-separated set is a finite collection of points \( \{x_j\} \) such that the sets \( B_{f,n}(x_j) \) are pairwise disjoint. Let \( N(f, n, \varepsilon) \) denote minimal cardinality of an \((n, \varepsilon)\)-net and \( S(f, n, \varepsilon) \) denote the maximal cardinality of an \((n, \varepsilon)\) separated set.

Define the topological entropy of \( f \) as:

\[
h_{\text{top}}(f) := \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log N(f, n, \varepsilon) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log S(f, n, \varepsilon) \]

Theorem 0.5. The topological entropy of a topologically mixing subshift of finite type \( \Sigma^A \) is the largest eigenvalue of \( A \).

Definition 0.6. Given a finite partition of a measure space \( (X, \mu) \) into measurable subsets \( \mathcal{P} = \{P_1, \ldots, P_n\} \), let \( H(\mathcal{P}) = \sum I(\mu(P)) \), where \( I(x) = -x \log x \). The wedge of partitions \( Q_1, \ldots, Q_m \) is denoted by \( Q_1 \vee \cdots \vee Q_m \), and the atoms of the partition are sets of the form \( Q_1 \cap \cdots \cap Q_m \), where \( Q_i \in Q_i \) for every \( i \). Given a \( \mu \)-preserving transformation \( T : X \to X \), let \( \mathcal{P}_n^T = \mathcal{P} \vee T^{-1}(\mathcal{P}) \vee \cdots \vee T^{-(n-1)}(\mathcal{P}) \).

The metric entropy or measure-theoretic entropy of \( T \) with respect to \( \mu \) is:

\[
h_{\mu}(f) := \sup_{\mathcal{P}} \lim_{n \to \infty} \frac{1}{n} H(\mathcal{P}_n^T). \]

A matrix \( B \) is called a stochastic matrix (subordinate to \( A \)) if all of its entries are nonnegative, the rows of \( B \) sum to 1, and \( B_{ij} = 0 \) if and only if \( A_{ij} = 0 \).

Theorem 0.7. Given a stochastic matrix \( B \) subordinate to \( A \), if \( \Sigma^A \) is topologically mixing, there exists a unique left eigenvector \( p \) of \( B \), and a \( \sigma \)-invariant probability measure \( \mu_B \) on \( \Sigma^A \) such that for every finite \( A \)-admissible word \( \omega = (\omega_1, \ldots, \omega_n) \), if \( C_\omega = \{(x_n) \in \Sigma^A : (x_n) \text{ begins with } \omega \} \), then

\[
\mu_B(C_\omega) = p_{\omega_1} \prod_{i=1}^{n-1} B_{\omega_i \omega_{i+1}}. \]

The metric entropy of \( \sigma \) with respect to \( \mu_B \) is \( \sum_{i,j} p_i I(B_{ij}) \).
Exercises

Problem 1. Consider the function \(d((x_n), (y_n)) = \left\{ \begin{array}{ll} 0, & x_n = y_n \text{ for every } n \\ 2^{-\ell}, & \text{where } \ell = \inf \{|n| : x_n \neq y_n\} \end{array} \right. \) Prove that \(d\) is a metric on \(\Sigma_{+} \) and \(\Sigma\). Describe \(B((x_n), \varepsilon)\) for an arbitrary sequence \((x_n)\). Use this description to show that \(d\) induces the product topology, when \(\Sigma_{+} \) and \(\Sigma\) are viewed as \(A^{\mathbb{N}_0}\) and \(A^{\mathbb{Z}}\), respectively. Show that the subshifts \(\Sigma^A \subset \Sigma \) and \(\Sigma^A_{+} \subset \Sigma_{+} \) are closed, \(\sigma\)-invariant subsets.

Problem 2. Assume that a 0-1 matrix \(A\) has no rows or columns which have all zeroes. Show that a subshift \(\Sigma^A\) is topologically transitive if and only if for every pair \(1 \leq i, j \leq m\) there exists some \(n \in \mathbb{N}\) such that \((A^n)_{ij} \neq 0\). Show that it is topologically mixing if and only if there exists some \(n \in \mathbb{N}\) such that every entry of \(A^n\) is nonzero.

Problem 3. Show that if \(f\) is an Anosov diffeomorphism of a compact manifold, then for sufficiently small \(\varepsilon\), \(\bigcap_{n \in \mathbb{N}} B_{f,n}(x, \varepsilon) = W^{st}_{\text{loc}}(x)\).

Problem 4. Let \(f : X \to X\) and \(g : Y \to Y\) be transformations of compact metric spaces. \(g\) is called a factor of \(f\) if there exists a continuous surjection \(\pi : X \to Y\) such that \(\pi \circ f = g \circ \pi\). Show that if \(g\) is a factor of \(f\), then \(h_{\text{top}}(g) \leq h_{\text{top}}(f)\). Furthermore, show that if there exists \(B\) such that \(|\pi^{-1}(x)| \leq B\), then \(h_{\text{top}}(g) = h_{\text{top}}(f)\).

Problem 5. Show that if \(A = \{0, 1\}\), then the linear expanding map on the circle \(L_2(x) = 2x \pmod{1}\) is a factor of \(\Sigma\) with factor map \(\pi((x_n)) = \sum_{n=0}^{\infty} 2^{-(n+1)} x_n\). Furthermore, show that \(\pi\) is one-to-one, with the exception of a countable subset, on which \(\pi\) is 2-to-1.

Problem 6. Show that if \(f : X \to X\) is a homeomorphism of a compact metric space, then \(h_{\text{top}}(f) = h_{\text{top}}(f^{-1})\).

Problem 7 (Baby Brin-Katok Theorem/Pesin Entropy Formula). * Assume that \(X\) is a compact Riemannian manifold, and \(f : X \to X\) is an Anosov diffeomorphism such that for every \(x \in X\), if \(v \in E^u_x\) then \(||df(v)|| = \lambda ||v||\) and if \(w \in E^s_x\), then \(||df(w)|| = \mu ||w||\) for some fixed \(\lambda > 1\) and \(\mu < 1\) independent of \(x\). Furthermore, assume that \(f\) preserves the Riemannian volume (i.e., \(f^* \text{vol} = \text{vol}\)). Show that there exist \(0 < c < C\) such that for every \(x \in X\), \(c \lambda^{dn} \leq \text{vol}(B_{f,n}(x, \varepsilon)) \leq C \lambda^{dn}\), where \(d = \dim(E^u)\). Conclude that the topological entropy of \(f\) is \(d \log \lambda\).

Problem 8. * Fix a length \(N\) and let \(\mathcal{L} \subset A^N\) be a language consisting of words of length \(N\). Say that a word \((x_n) \in \Sigma\) is \(\mathcal{L}\)-admissible if for every \(m\), \((x_m, \ldots, x_{m+n-1})\) is in \(\mathcal{L}\). Let \(\Sigma^\mathcal{L}\) denote the set of infinite, \(\mathcal{L}\)-admissible sequences, and show that \(\mathcal{L}\) is shift-invariant and closed.

Now consider the set \(\mathcal{L}\) as the alphabet to define a shift, so that the letters of the alphabet are the collection \(\mathcal{L}\) of words of length \(N\) in \(A\). Define the adjacency matrix \(A(\mathcal{L})\) such that if \(\sigma_1, \sigma_2 \in \mathcal{L}\), then \(A_{\sigma_1, \sigma_2} = 1\) if and only if the last \(N - 1\) letters of \(\sigma_1\) and first \(N - 1\) letters of \(\sigma_2\) coincide. Show that the shifts on \(\Sigma^A(\mathcal{L})\) and \(\Sigma^\mathcal{L}\) are topologically conjugate. Moral: All finite language subshifts can be studied using 2-step subshifts.

Problem 9. * Prove that the global stable and unstable manifolds form a foliation using the following scheme:

1. Show that if \(W^{st}_{\text{loc}}(x_1) \cap W^{st}_{\text{loc}}(x_2) \neq \emptyset\), then their union is a connected manifold of the same dimension (Use the uniqueness and dynamical characterization).
2. Say that \(W^s(x) = \{y \in X : d(f^n(x), f^n(y)) \to 0\}\). Show that if \(z \in W^s(x)\), then \(W^s_{\text{loc}}(z) \subset W^s(x)\). Define a topology on \(W^s(x)\) in which \(d(z_1, z_2) < \varepsilon\) if and only if \(z_1 \in W^s_{\text{loc}}(z_2)\) and the points are \(\varepsilon\)-close on the local stable manifold.
3. Prove that \(W^s(x)\) is an immersed submanifold without self-intersections.
(4) Given \( x \in X \), define a map \( \psi_x : W^s_{\text{loc}}(x) \times W^u_{\text{loc}}(x) \rightarrow Y \) by \((y,z) \mapsto W^u_{\text{loc}}(y) \cap W^s_{\text{loc}}(x)\).

Prove that the map is well-defined and forms a foliation atlas, so that each \( W^u \) and \( W^s \) are continuous foliations with \( C^r \)-leaves.