UNIFORM HYPERBOLICITY, COCYCLES AND RIGIDITY - WEEK 2

Definitions and Main Theorems

Definition 0.1. A diffeomorphism $f : X \to X$ is called *Anosov* if there exists some $C > 0$, some $0 < \lambda < 1$ and at every $x \in X$, a splitting of the tangent bundle $TX = E^s_x \oplus E^u_x$ such that:

(1) $df(E^s_x) = E^s_{f(x)}$ and $df(E^u_x) = E^u_{f(x)}$,
(2) if $v \in E^s_x$, then $||df^n(v)|| \leq C\lambda^n \, ||v||$, and
(3) if $v \in E^u_x$, then $||df^{-n}(v)|| \leq C\lambda^n \, ||v||$.

Example 0.2. *If $A \in SL(n, \mathbb{Z})$, the map $f : \mathbb{T}^n \to \mathbb{T}^n$ defined by $f(\bar{x}) = \bar{A}x$ is well-defined (where $x \in \mathbb{R}^n$ and $\bar{x}$ represents the equivalence class in $\mathbb{T}^n$). f is called the toral automorphism induced by A, and is Anosov if and only if A has no eigenvalues on $S^1 \subset \mathbb{C}$. In this case, we call it a hyperbolic toral automorphism.*

Definition 0.3. A homeomorphism $f : X \to X$ is called *topologically mixing* if for any pair of nonempty open sets $U, V \subset X$, there exists $N \in \mathbb{N}$ such that if $n \geq \mathbb{N}$, then $f^n(U) \cap V \neq \emptyset$.

Theorem 0.4. *Hyperbolic toral automorphisms are topologically mixing.*

Definition 0.5. Let $f : X \to X$ be a homeomorphism of a metric space. An ε *pseudo-orbit* is a sequence (x_i), where $i \in [N_1, N_2] \cap \mathbb{Z}$ (including the possibility that $N_1 = -\infty$ and $N_2 = \infty$) such that for all $N_1 \leq i \leq N_2$, $d(f(x_i), x_{i+1}) < \varepsilon$. The pseudo-orbit is *periodic* if $N_1 = 0$, $0 < N_2 < \infty$ and $d(f(x_{N_2}), x_0) < \varepsilon$. We call $N_2 + 1$ its period.

Definition 0.6. A homeomorphism $f : X \to X$ is said to *satisfy the shadowing property* if for any $\varepsilon > 0$, there exists $\delta > 0$ such that if (x_i) is a δ pseudo-orbit, there exists a point $x \in X$ such that $d(f^i(x), x_i) < \varepsilon$.

A homeomorphism $f : X \to X$ is said to *satisfy the closing lemma* if for any $\varepsilon > 0$, there exists $\delta > 0$ such that if (x_i) is a periodic δ psuedo-orbit, there exists a periodic point $x \in X$ with the same period as the psuedo-orbit such that $d(f^i(x), x_i) < \varepsilon$.

Exercises

Problem 1. Prove Dirichlet's Approximation Theorem: If $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, then for every $\varepsilon > 0$, there exists $q \in \mathbb{N}$, $p \in \mathbb{Z}$ such that $|q\alpha - p| < \varepsilon$.

Problem 2. Let X be a compact metric space, $f : X \to X$ be a homeomorhism, and $\varphi_t : \tilde{X} \to \tilde{X}$ be the special flows over f with an arbitrary roof function r. Show that f is topologically transitive if and only if φ_t is topologically transitive.

Problem 3. Formulate a criterion for topologically transitivity of continuous flows without open orbits using open sets, and prove that it is equivalent to the flow having a dense orbit.

Problem 4. Show that a suspension flow (ie, a flow with a constant-time roof function) is never topologically mixing. [*Remark*: The same is *not* true if the roof function is allowed to vary!]

Problem 5. Show that expanding maps of $\mathbb{R}/\mathbb{Z}$ are topologically mixing.

Problem 6. Show that if $A \in SL(n, \mathbb{Z})$ does not have any roots of unity as eigenvalues, then a point $x \in \mathbb{T}^n$ is periodic for the induced toral automorphism if and only if x is periodic.

[*Hint:* For one direction, write $x = (p_1/q, p_2/q, \dots, p_n/q)$. Show that $f(x)$ is written in the same form, and use the pidgeon hole principle. For the other direction, lift to $\mathbb{R}^n$, and show that $A^k - \mathrm{Id}$ is invertible under the assumption on A.]

Problem 7. Show that the shadowing property and closing lemma are invariants of topological conjugacy. Conclude that if X is a compact metric space, these properties do not depend on the metric that X carries, if it induces the same topology.

Problem 8. Show that linear expanding maps of the circle satisfy the closing lemma. [*Hint*: Use the lift to $\mathbb{R}$, and consider only the first and last point of the pseudo-orbit]

Problem 9. Show that if $\alpha \in \mathbb{R}^n$, then $T_\alpha : \mathbb{T}^n \to \mathbb{T}^n$ does *not* satisfy the shadowing property or closing lemma.