0) SET Starter Kit

1) Linear Algebra.

2) ~~Stat~~ ALE's.

3) Statements Rank 1 Osel.

4) Pf sketch

5) Statement Higher Rank Osel.

$\boxed{10:45 - 12:05}$

Oseledets: Let X be metric, $(\mu, f) : X \circlearrowleft$ be ergodic,

(noninvertible)

$\pi : V \to X$ be a measurable vector bundle,
finite rank.

$F \in \cancel{\mathrm{End}}(V; f)$ be an extension.

$\boxed{\begin{array}{l} \pi \circ F = f \circ \pi, \\ \forall x \in X: \\ F_x = F|_{V_x} : V_x \to V_{f(x)}. \end{array}}$

Let $|\cdot|$ be a measurable fiberwise norm on V such that

$$\left[x \longmapsto \log^+\left(|F_x : V_x \to V_{f(x)}|\right)\right] \in L^1(\mu).$$

Then ~~$\exists \ell, \exists x_1$~~

$\exists \ell \in \{1, 2, \ldots, \mathrm{rank}(V)\}$,

$\exists x^1, x^2, \ldots, x^\ell \in [-\infty, \infty[$

with ~~$x^1 > x^2 > \ldots > x^\ell$~~ $-\infty \leq x^\ell < \ldots < x^2 < x^1$

$\exists (F, f)$-invariant measurable subbundles

(forward Oseledets flag) $\to$ $0 \lneq V^{\leq x^\ell} \lneq V^{\leq x^{\ell-1}} \lneq \ldots \lneq V^{\leq x^1} = V$

such that $\forall v \in V^{\leq x^i} \setminus V^{\leq x^{i+1}}$: $\lim_{n \to \infty} \frac{\log |F^n_x v|}{n} = x^i$

linear multiplicity of $x^i = L_M(x^i)$

$$= \dim\left(V^{\le x^i}\right) - \dim\left(V^{\le x^{i-1}}\right).$$

Oseledets (~~Noninve~~ Invertible): f (f, F) invertible,

applying Noninvertible Oseledets to

(F^{-1}, f^{-1}) gives

$$k \ \ell^{*}, \quad \eta^1, \eta^2, \ldots, \eta^k$$

$$-\infty \le \eta^k < \cdots < \eta^2 < \eta^1,$$

$\exists\, (F^{-1}, f^{-1})$-inv. measurable subbundles

(backward Oseledets filtration) $$0 \subsetneq V^{\le \eta^k} \subsetneq V^{\le \eta^{k-1}} \subsetneq \cdots \subsetneq V^{\le \eta^1} = V$$

such that

$$\forall v \in V^{\leq \eta^j} \setminus V^{\leq \eta^{j-1}}:$$

$$\lim_{n \to \infty} \frac{\log |F^{-n}_v|}{n} = \eta^j$$

$$\eta^j = \lim_{n \to \infty} \frac{\log |F^{-n}_v|}{n} = -\lim_{n \to \infty} \frac{\log |F^{-n}_v|}{-n}$$

$$= -\lim_{n \to \infty} \frac{\log m(F^{n}_v)}{n}$$

$$-\eta^j = \lim_{n \to -\infty} \frac{\log |F^{n}_v|}{n} = \lim_{n \to \infty} \frac{\log m(F^{n}_v)}{n}.$$

On a full ~~set~~ measure set, ~~compat~~ ~~compatible~~ forward and backward Oseledets coincide

$$\Rightarrow \boxed{l = k}$$

$$-\infty \leq \eta^l < \chi^{l-1} < \cdots < \chi^1$$

$$-\infty \leq \eta^l < \eta^{l-1} < \cdots < \eta^1$$

$$-\eta^1 < -\eta^2 < \cdots < -\eta^l \leq \infty.$$

$$\chi^1 = -\eta^l$$

$$\chi^2 = -\eta^{l-1}$$

$$\vdots$$

$$\chi^l = -\eta^1$$

$$\boxed{-\eta^j = \chi^{l-j+1}}$$

Define $V^{x^1} = V^{\leq x^1} \cap V^{\leq \eta^\ell}$

$V^{x^2} = V^{\leq x^2} \cap V^{\leq \eta^{\ell-1}}$

$\vdots$

$V^{x^\ell} = V^{\leq x^\ell} \cap V^{\leq \eta^1}$

Then $V = \bigoplus_{i=1}^{\ell} V^{x^i}$ is an (F,f)-inv.

measurable decomposition,

$\forall v \in V \setminus 0$:

$$v \in V^{x^i} \iff \lim_{|n| \to \infty} \frac{\log |F^n_v|}{n} = x^i.$$

$\underline{Ex}$: $\mathrm{rank}(V) = 1 \Rightarrow$ Oseledets $=$ Birkhoff.

$$(F, f) : [V \to X] \circlearrowleft .$$

$\forall v \in V_x$.

$$\phi : X \longrightarrow \mathbb{R}$$
$$x \longmapsto \log |F_x| \cancel{\frac{|F_x v|}{|v|}}$$

$$\phi(x) = \log \frac{|F_x v|}{|v|}$$

$\boxed{\text{Birkhoff} \Rightarrow \text{Osel}}$

$$\phi^+ = \max\{\phi, 0\} \in L^1(\mu)$$

then log Birkhoff for $\phi^+ \in L^1)$

$$\cancel{\frac{1}{n} \sum_{k=0}^{n-1} \phi \circ f(x) = \log}$$

$v \in V_x$ $F_x v \in V_{f(x)}$

$$\phi \circ f(x) = \log |F_{f(x)}| = \log \frac{|F_{f(x)} F_x v|}{|F_x v|}$$

$$= \log \left(\frac{|F_x^2 v|}{|F_x v|} \right) . \Rightarrow \frac{1}{n} \sum_{k=0}^{n-1} \phi \circ f(x) = \frac{1}{n} \log \left(\frac{|F_x v|}{|v|} \frac{|F_x^2 v|}{|F_x v|} \right)$$

$$= \frac{1}{n} \log |F_x^n| \to \mathbb{E}_\mu(\phi) = \chi^1 .$$

$\boxed{\text{Oseledets} \Rightarrow \text{Birkhoff}}$ $\quad \phi \in L^1(\mu)$.

$$F : X \times \mathbb{R} \to X \times \mathbb{R}$$
$$(x, v) \mapsto \left(f(x), e^{\phi(x)} v \right)$$

$\cancel{\frac{\log |F_x^n v|}{n}}$ $\quad \cancel{F_x^n v}$

$$F^n(x, v) = \left(f^n(x), e^{\sum_{k=0}^{n-1} \phi \circ f^k(x)} v \right)$$

$$\frac{\log |F_x^n v|}{n} = \frac{1}{n} \sum_{k=0}^{n-1} \phi \circ f^k(x).$$

$$\longrightarrow \chi^1.$$

Pf of Noninvertible Oseledets:

Lemma A: At least one of the following holds:

ULEX

(I) $\exists!\,\chi\in\mathbb{R},\ \forall x\in_\mu X,\ \forall v\in V_x\setminus 0$:

$$\lim_{n\to\infty}\frac{\log|F_x^n v|}{n}=\chi.$$

(uniformly in v)

(II) $\exists\, 0\subsetneq W\subsetneq V$ (F,f) invariant.

Lemma B: Let $0\to W\to V\xrightarrow{p} U\to 0$

be an s.e.s. of extensions over (μ,f).

If W and U have the SLE condition (single Lyap. exponent), and

$\chi^{W} > \chi^{U} \geq -\infty$ (maximal growth occurs in a proper subbundle),

then the s.e.s. splits, i.e.

$\exists\ \sigma : U \to V$ linear injective, (F, f)-invariant :

$$V = W \oplus \sigma(U), \quad p \circ \sigma = \mathrm{id}_U .$$

(F, f)-inv.

$$F_x \circ \sigma_x = \sigma_{f(x)} \circ F_{U,x}$$

$$\begin{tikzcd}
U_x \arrow[r, hook, "\sigma_x"] \arrow[d, "F_{U,x}"'] & V_x \arrow[d, "F_x"] \\
U_{f(x)} \arrow[r, hook, "\sigma_{f(x)}"] & V_{f(x)}
\end{tikzcd}$$

Pf of Oseledets Noninv. :

By induction on Rank(V)

- Rank(V) = 1 by Birkhoff (for $\phi^{+} \in L^{1}$)
- Suppose statement holds $\forall$ rank(V) $\leq n-1$.

Let rank(V) = n.

By Lemma A,

V has SLE

$\Downarrow$

Done.

OR

$\exists\ 0 \underset{\neq}{<} W \underset{\neq}{<} V$

(F, f)-inv.

Pick a $0 \subsetneq W \subsetneq V$ with minimal codimension.

$\Rightarrow$ V/W has SLE, say $\boxed{x^{V/W}}$

rank$(W) < n$, hence by induction assumption,

$\exists$ ~~x^1 s.t.~~

$$-\infty \le x^{\ell} \le \cdots < x^{1},$$

$$0 < W^{\le x^{\ell}} < W^{\le x^{\ell-1}} < \cdots < W^{\le x^{2}} < W^{\le x^{1}} = W$$

$\boxed{x^1 < x^{V/W}}$ XOR $\boxed{x^1 \geq x^{V/W}}$

Done.

$$W^{\le x^2} < W^{\le x^1} = W < V$$

$$0 \to W^{\le x^1} / W^{\le x^2} \to V / W^{\le x^2} \to V/W \to 0$$

$$x^{W^{\le x^1}/W^{\le x^2}} = x^1 > x^{V/W}.$$

$\Rightarrow$ By Lemma B,

$\exists$ linear inj: $\sigma : V/W \to V / W^{\le x^2}$:

$$V / W^{\le x^2} = W / W^{\le x^2} \oplus \sigma\left(V/W\right)$$

(F, f)-inv.

$$\left[\begin{array}{l} \text{preimage } \left(\sigma\left(V/W\right)\right) \\ \text{under } V \to V/W^{\le x^2} \end{array}\right]$$

$$=: V^{\le x^2}$$

Then (1) Any vector outside $V^{\le x^2}$ grows at rate x^{1}

(2) $V^{\le x^2}$ maps naturally to V/W with $LE = x^{V/W}$

V has ULEX

OK

$\exists$ (Ff)-inv.

$0 \lneq W \lneq W$

W be with minimal codim.

$\Downarrow$

V/W has ULEX

$$-\infty \le x^{\ell}(W) < \cdots < x^{1}(W)$$

$$0 < W^{\le x^{\ell}} < \cdots < W^{\le x^{1}}$$

$x^{1}(W) \le x(V/W)$ XOR

$x^{1}(W) > x(V/W)$

Done.

$$-\infty \le x^{\ell}(W) < \cdots < x^{1}(W) \le x(V/W)$$

$$x^{\ell+1}(V) \quad \cdots \quad x^{2}(V) \quad x^{1}(V)$$

$$\boxed{x^2(W) < x\left(V/W\right) \le x^1(W)} \qquad X \text{ OK} \qquad \boxed{x\left(V/W\right) \le x^2(W) < x^1(W)}$$

$$0 \to W/W^{\le x^2} \to V/W^{\le x^2} \to V/W \to 0$$

$$x\left(W/W^{\le x^2}\right) = x^1(W) \geq x\left(V/W\right)$$

By Lemma B, $\exists$ linear inj. $\sigma : V/W \to V/W^{\le W^2}$:

$$V/W^{\le x^2} = \left(W/W^{\le x^2}\right) \oplus \sigma\left(V/W\right). \quad (F,f)\text{-inv.}$$

$$p^{\le x^2} : V \to V/W^{\le x^2}. \qquad W^{\le x^2} \to V \to V/W^{\le x^2}$$

Put $V^{\le x^2} = \left(p^{\le x^2}\right)^{-1}\left(\sigma\left(V/W\right)\right)$

Pf of Lemma A :

$$\phi : \mathbb{P}(V) \to \mathbb{R}$$
$$[v] \longmapsto \log\left(\frac{|Fv|}{|v|}\right).$$

$$\mathbb{E}_{\bullet}(\phi) : \mathcal{M}_1(\mathbb{P}(V), \mu) \to \mathbb{R}$$ affine, C^0.

$$\hat{\mu} \in \operatorname{argmin}\left(\mathbb{E}_{\bullet}(\phi)\right), \quad \chi = \mathbb{E}_{\hat{\mu}}(\phi).$$

BET $\Rightarrow$ $\forall$ $\hat{\mu}$-a.e. $[v] \in \mathbb{P}(V)$:

$$\frac{1}{n} \sum_{k=0}^{n-1} \phi \circ F^k [v] \to \mathbb{E}_{\hat{\mu}}(\phi)$$

$$\| $$

$$\log \frac{|F^n v|}{|v|} = \log |F^n v| - \log |v|$$

$W = \mathrm{Span}\{ v \mid \frac{1}{n}\sum_{i=0}^{n-1} \phi \circ F^i[v] \to \cancel{x} \}.$

Then $W \neq 0$, W has ULEX;

W is (F, ϕ)-inv.

$\square$.

Pf of Lemma B :

Have : $0 \to W \to V \xrightarrow{p} U \to 0$

SES of extensions over (F,f) AE,

W, U ULEX,

$$\chi(W) > \chi(U) \geq -\infty.$$

Want : $\sigma : U \to V$: $V = W \oplus \sigma(U)$, (F,f)-inv

Want : $\sigma(U)$ (F,f)-inv.

Fix a linear splitting $\sigma_0 : U \to V$.

$V = W \oplus \sigma_0(U)$, possibly not (F,f)-inv.

$\sigma = \sigma_0 + \tau$, where $\tau : U \to W$.

In the splitting

$$F_x : W_x \oplus \sigma_0(U_x) \to W_{f(x)} \oplus \sigma_0(U_{f(x)}),$$

$$F_x = \begin{pmatrix} A_x & B_x \\ C_x & D_x \end{pmatrix}.$$

$$A_x = F_x|_{W_x} : W_x \to W_{f(x)}$$

$$B_x : \sigma_0(U_x) \to W_{f(x)}.$$

$$C_x = 0$$

$$D_x = F_x|_{\sigma_0(U_x)} : \sigma_0(U_x) \to \sigma_0(U_{f(x)}).$$

~~Want~~: ~~e~~ Consider the graph of τ.

$$\boxed{\begin{aligned} \underline{\text{Rem}}: \quad & w + \sigma_0(u) \in \sigma(U) \\ \Longleftrightarrow \quad & w = \tau(u) \end{aligned}}$$

$$\begin{pmatrix} A_x & B_x \\ 0 & D_x \end{pmatrix} \begin{pmatrix} \tau_x(u) \\ u \end{pmatrix} = \begin{pmatrix} \tau_{f(x)}(\bar{u}) \\ \bar{u} \end{pmatrix}.$$

$$A_x \tau_x(u) + B_x(u) = \tau_x(\bar{u})$$

$$D_x u = \bar{u}$$

$$\boxed{A_x \circ \tau_x + B_x = \tau_{f(x)} \circ D_x}$$

$$\tau_x = A_x^{-1} \circ \tau_{f(x)} \circ D_x - A_x^{-1} \circ B_x$$

$$\tau_x = A_x^{-1} \circ \tau_{f(x)} \circ D_x - A_x^{-1} \circ B_x .$$

$$= A_x^{-1} \circ \left(A_{f(x)}^{-1} \circ \tau_{f^2(x)} \circ D_{f(x)} - A_{f(x)}^{-1} \circ B_{f(x)} \right) \circ D_x$$

$$- A_x^{-1} \circ B_x$$

$$= A(2,x)^{-1} \circ \tau_{f^2(x)} \circ D(2,x)$$

$$- \Big[A(2,x)^{-1} \circ B_{f(x)} \circ D_{\cancel{x}}(1,x)$$

$$+ A(1,x)^{-1} \circ B_x \circ D(0,x) \Big]$$

$$= \dots$$

$$= \mathcal{A}(n,x)^{-1} \circ \tau_{f^n(x)} \circ \mathcal{D}(n,x)$$

$$- \Big[\mathcal{A}(n,x)^{-1} \circ B_{f^{n-1}(x)} \circ \mathcal{D}(n-1,x) + \dots + \mathcal{A}(1,x)^{-1} \circ B_x \circ \mathcal{D}(0,x) \Big]$$

$$= \overbrace{\mathcal{A}(n,x)^{-1} \circ \tau_{f^{n}(x)} \circ \mathcal{D}(n,x)}^{\to 0}$$

$$- \underbrace{\sum_{k=0}^{n-1} \mathcal{A}(k-1,x)^{-1} \circ \mathcal{B}_{f^{k}(x)} \circ \mathcal{D}(k,x)}_{\to \text{ converge.}}$$

$$\mathcal{X}(w) > \mathcal{X}(u)$$

$$0 > \mathcal{X}(u) - \mathcal{X}(w).$$

$\square$.