G&T topics Sp2025 - Problem Set 2

(1) A group action $G \curvearrowright X$ is called *group transitive* if $G \cdot x = X$ for some $x \in X$.
  (a) Show that if $H \subset G$ is a closed subgroup, then the left-translation action $G \curvearrowright G/H$ is a group transitive action, and every C^∞ group transitive action is C^∞ conjugated to such an action.
  (b) Show that a group transitive action is locally free if and only if H is discrete, where H is as above.
  (c) (Liouville theorem) Show that if $\mathbb{R}^k \curvearrowright X$ is a locally free C^∞ group transitive action, then X is a k-dimensional torus.
  (d) Show that any cocycle $\varphi : \mathbb{R}^k \times X \to \mathbb{R}$ over a locally free group transitive action is cohomologous to a constant.
  (e) Give an example of a group transitive action that has a cocycle which is not cohomologous to a constant.

(2) Show that if $G \curvearrowright X$ is a group action, and μ is a G-invariant measure, and $\varphi : G \times X \to \mathbb{R}$ is a cocycle, then the map

$$\Phi_\mu(g) = \int_X \varphi(g, x)\, d\mu$$

is a homomorphism from G to $\mathbb{R}$. Furthermore, show that if φ is cohomologous to a constant cocycle, then Φ_μ is independent of μ. Does this hold for matrix-valued cocycles?

(3) Show that every orientable principal $\mathbb{R}$-bundle has a continuous section.

(4) Define $\beta(v, w) = v^T \begin{pmatrix} 0 & \mathrm{id} & 0 \\ \mathrm{id} & 0 & 0 \\ 0 & 0 & \mathrm{id} \end{pmatrix} w$, hwere the matrix is a block square matrix with blocks of size $(p, p, q-p)$. Show that β has signature (q, p).

(5) Verify computationally that $[\mathfrak{g}_r, \mathfrak{g}_{r'}] \subset \mathfrak{g}_{r+r'}$ for any roots r and r' of the Lie algebra $\mathfrak{so}(p, q)$ (You can use the matrix form of the Lie algebra above)

(6) Identify an $\mathbb{R}$-split Cartan subalgebra for the standard bilinear form of signature (q, p): $\beta(v, w) = \sum_{i=1}^q v_i w_i - \sum_{i=q+1}^{q+p} v_i w_i$.

(7) Recall that if $\mathfrak{a} \subset \mathfrak{g}$ is an $\mathbb{R}$-split Cartan subalgebra of a semisimple Lie algebra $\mathfrak{g}$, and $X \in \mathfrak{a}$, then

$$E^u_X = \bigoplus_{r \in \Delta : r(X) > 0} \mathfrak{g}_r.$$

Show that if $X_1, \dots, X_\ell \in \mathfrak{a}$, find a characterization of $E^u_{X_1,\dots,X_\ell} := \bigcap_{i=1}^\ell E^u_{X_i}$. Show that for every root r, there exists $X_1, \dots, X_\ell$ such that $E^u_{X_1,\dots,X_\ell} = \oplus_{t>0} \mathfrak{g}_{tr}$.

(8) Show that if $\mathfrak{a}$ is an $\mathbb{R}$-split Cartan subalgebra, and $Z_\mathfrak{g}(\mathfrak{a}) = \mathfrak{a} \oplus \mathfrak{m}$ is the sum of $\mathfrak{a}$ and a compact subalgebra $\mathfrak{m}$, then for every $Y \in \mathfrak{m}$, ad_Y preserves the root space decomposition and has purely imaginary eigenvalues.

(9) Show that $Z_\mathfrak{g}(\mathfrak{a}) \oplus E^u_X$ is a subalgebra for every $X \in \mathfrak{a}$, and solvable when $\mathfrak{m}$ is abelian.

(10) Call an action $\alpha : H \curvearrowright X$ *uniformly locally free* if there exists a neighborhood $U \subset H$ containing $e \in H$ such that for every $x \in X$, $g \mapsto \alpha(g)x$ is injective from U. Call an action *pointwise locally free* if for every $x \in X$, $\mathrm{Stab}_\alpha(x)$ is a discrete subgroup of H.

(a) Show that every uniformly locally free action is pointwise locally free.
(b) Find an example of a uniformly locally free action which is not free.
(c) Find an example of an action which is pointwise locally free, but not uniformly locally free.
(d) Let G be a Lie group, and $H, Q \subset G$ be closed subgroups of G. Show that the translation action $\alpha : H \curvearrowright G/Q$ is pointwise locally free if and only if $H \cap gQg^{-1}$ is discrete in H for all $g \in G$.
(e) * Let G be a semisimple Lie group, $\Gamma \subset G$ be a lattice, and $\varphi_t : G/\Gamma \to G/\Gamma$ be the homogeneous flow $\varphi_t(g\Gamma) = \exp(tX)g\Gamma$. Show that if φ_t is not uniformly locally free, then Γ is not cocompact, and $X \in \mathrm{Lie}(G)$ is not $\mathbb{C}$-semisimple (ie, ad_X has a nontrivial Jordan block).