G&T topics Sp2025 - Problem Set 1

(1) Show that if ω_1 and ω_2 are right Haar forms on a Lie group G, then $\omega_1 = \lambda\omega_2$ for some $\lambda \in \mathbb{R}$.

(2) Show that if ω is a right Haar form and $g \in G$, then $(L_g)^*\omega$ is a right Haar form.

(3) Show that if ω is a right Haar form, the map $\lambda : G \to \mathbb{R}$ defined by

$$\lambda(g) = \frac{(L_{g^{-1}})^*\omega}{\omega}.$$

is a well-defined homomorphism independent of the choice of ω.

(4) Show that if G has a lattice, then it is unimodular.

(5) Let X be an $n \times n$ matrix, and consider the Lie group $H = \mathbb{R} \ltimes_X \mathbb{R}^n$ whose multiplication is given by

$$(t, v) * (s, w) = (t + s, v + \exp(tX)w).$$

(a) Show that this multiplication makes H a Lie group.

(b) Identify a basis of the Lie algebra, and find the Lie brackets of the basis elements.

(c) Show that H is solvable, and nilpotent if and only if the only eigenvalue of X is 0.

(d) Find a matrix group isomorphic to H.

(e) Show that H is unimodular if and only if $\mathrm{Tr}(X) = 0$ (ie, $\det(\exp(X)) = 1$).

(6) Show that the two suspension constructions described in class are smoothly conjugated (try it for suspending a $\mathbb{Z}$-action first if you're stuck).

(7) Let G be a Lie group, and let

$$\phi : TG \to G \times T_eG \qquad \phi(v; g) = (g, dR_{g^{-1}}(v))$$

Show that ϕ is a diffeomorphism, and that $\phi(dR_g(v; h)) = (v; hg)$. Induce a trivialization of $T(G/\Gamma)$.

(8) Show that the group

$$H = \begin{pmatrix} 1 & * & * & \dots & * \\ 0 & 1 & * & \dots & * \\ 0 & 0 & \ddots & \dots & * \\ 0 & 0 & \dots & 1 & * \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

is nilpotent.

(9) Show that if H is a connected solvable Lie group, then $[H, H]$ is a nilpotent Lie group (you may use the theorems of Lie and Engel).

(10) Show that for a Lie algebra $\mathfrak{g}$, the Killing for B is bilinear, symmetric and invariant under automorphisms of $\mathfrak{g}$. Compute the Killing form for $\mathbb{R} \ltimes_X \mathbb{R}^2$, $\mathfrak{so}(3)$ and $\mathfrak{sl}(2, \mathbb{R})$