KV- Topics
3/6/25

<u>Def</u>: If V is a variety, a function $f : \mathbb{R}^k \to V$ is called <u>essentially rational</u> if $\exists$ rational $R : \mathbb{R}^k \to V$ such that $f = R$ on a set of full Lebesgue measure.

(Margulis trick).

<u>Theorem</u> (Margulis; <u>3.4.4 in Zimmer</u>):
If $f : \mathbb{R}^k \times \mathbb{R}^m \to \mathbb{R}$ is measurable, for leb-ae $x \in \mathbb{R}^k$, $f_x(y) = f(x,y)$ is essentially rational, and for leb-ae $y \in \mathbb{R}^m$, $f^y(x) = f(x,y)$ is essentially rational, then f is essentially rational.

Pf: By induction, it suffices to consider the case $m = 1$.

Reduction: It suffices to show that

$\exists A \subseteq \mathbb{R}^k$: $leb_k(A) > 0$ and a rational function $R : \mathbb{R}^k \to \mathbb{R}$ such that $f = R$ æ on $A \times \mathbb{R}$.

Pf of Reduction: For æ $y \in \mathbb{R}$, $R^y = f^y$ on A by Fubini.

Since f^y is rational, $f^y = R^y$ on $\mathbb{R}^k$ for æ y.

$\Rightarrow$ $f(x,y) = R(x,y)$ æ x & æ y

$\overset{\text{(Fubini)}}{\Leftrightarrow}$ æ (x,y).

$\hookrightarrow$

Fix $A_{r,s} = \left\{ x \in \mathbb{R}^k \;\middle|\; f_x(y) = \dfrac{P_x(y)}{q_x(y)} \;\; \begin{array}{l} \deg(P_x) = r \\ \deg(q_x) = s \\ \text{for a.e. } y \end{array} \right\}$

By assumption, $\bigsqcup_{r,s \in \mathbb{N}} A_{r,s}$ has full measure

$\Rightarrow \exists (r_0, s_0)$ such that $A_{r_0,s_0} =: A$ has positive measure in $\mathbb{R}^k$.

Let $B = \left\{ y \in \mathbb{R} \;\middle|\; f_x(y) = \dfrac{P_x(y)}{q_x(y)} \text{ for a.e. } x \in A \right\}$

By Fubini, B has full measure in $\mathbb{R}$.

Choose $t_0, \ldots, t_{r_0+s_0} \in B$ such that $g_j := f^{t_j} : \mathbb{R}^k \to \mathbb{R}$ are essentially rational.

$\hookrightarrow$

For $x \in A$, write

$$f(x,y) = f_x(y) = \frac{P_x(y)}{q_x(y)}$$

$$= \frac{y^{r_0} + a_{r_0-1}(x)\, y^{r_0-1} + \dots + a_0(x)}{b_{s_0}(x)\, y^{s_0} + b_{s_0-1}(x)\, y^{s_0-1} + \dots + b_0(x)}$$

Goal: Show that $a_i(x), b_i(x)$'s coincide with rational functions on A.

Note: essentially rational

$$c_j(x) = f^{t_j}(x) = f(x, t_j)$$

$$= \frac{t_j^{r_0} + \dots + a_0(x)}{b_{s_0}(x)\, t_j^{s_0} + \dots + b_0(x)}$$

for ae $x \in \mathbb{R}^k$.

$$\Rightarrow \left[c_j(x)\, t_j^{s_0}\right] b_{s_0}(x) + \cdots + \left[c_j(x)\right] b_0(x)$$

$$- \left[t_j^{r_0-1}\right] a_{r_0-1}(x) - \cdots - a_0(x) = t_j^{r_0}$$

Consider this as a linear equation in the unknowns $a_i(x)$, $b_i(x)$ and coefficients t_j, $c_j(x)$.

Since inversion is a rational function on the space of matrices (Cramer's Rule), a_i and b_i are essentially rational in $t_j, c_j(x)$

$\Rightarrow$ a_i, b_i are essentially rational on A.

$\square$.

Thm* (Journé): Assume $f: (0,1)^k \times (0,1)^m \to \mathbb{R}$ has the property that $f_x = f(x,\cdot)$ belongs to a compact subset of

$$C^{r,\theta}\left((0,1)^k; \mathbb{R}\right)$$

and similarly for $f^y = f(\cdot, y)$. Then $f \in C^r$.

On the other hand, $\exists$ functions:

$f(x,\cdot)$ and $f(\cdot, y)$ are C^r but f is not C^r.

Exr: Show that Margulis' thm holds for $f : \mathbb{R}^k \times \mathbb{R}^m \to V$, V a variety.

- if V affine, use coordinate functions.
- if V projective, use that V is covered by finitely many affine charts.

Cor: Let U_+ be the unstable subgroup of some generic $a \in A \subseteq G$, A Cartan.

Then if $f : U_+ \to V$ has that $f|_{U_x u}$ is essentially rational $\forall x \in \Delta_+$ and $u \in U_+$, then f is essentially rational.

Pre-Lemma: Each U_x is rationally equivalent to ~~$\mathbb{R}$~~ $\mathbb{R}^{\dim(U_x)}$.

Pf of pre-lemma:

$\exp : \mathrm{Lie}(U_x) \to U_x$ is a

$\mathrm{Lie}(U_x) \cong \mathbb{R}^{\dim(U_x)}$

polynomial since U_x is nilpotent (similarly log is polynomial).

Things to check:

- exp is onto
- U_x is simply connected (exp is injective)

$\hookrightarrow$

$\boxed{\text{exp is onto}}$ BCH.

$$\exp(X)\exp(Y) = \exp\big(P(X,Y)\big)$$

power series in general; polynomial for nilpotent.

$\boxed{\mathcal{U}_x \text{ is simply connected}}$

Assume otherwise. Then $\exists X \in \mathrm{Lie}(\mathcal{U}_x)$ such that

$\exp(t_0 X) = e$ for some $t_0 \in \mathbb{R}$

Choose $Y \in \mathrm{Lie}(A)$ s.t. $x(Y) < 0$

$$\Rightarrow \exp(sY)\exp(t_0 X)\exp(-sY) = e$$

$$\|$$

$$\exp\big(\mathrm{Ad}(\exp(sY))\, t_0 X\big)$$

$$= \exp\left(e^{s\,x(Y)} t_0 X\right)$$

Rem:

- $\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$ is not in the image of exp, but $\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \in SL(2,\mathbb{R})$

$\Rightarrow \exp(tX) = e, \ \forall t,$

a contradiction.

Last thursday: U_+ is built in the following way:

$$U_+ = U_{[\ell]}$$

$$1 \to U_{[k]} \to U_{[k+1]} \to U_k \to 1.$$

$$U_{[k+1]} \overset{nat}{\cong} U_{[k]} \times U_{k+1}$$

Pf of Cor:

Since each $U_\ell \overset{rat}{\cong} \mathbb{R}^{\dim(U_\ell)}$

we can inductively apply Margulis lemma to get f is essentially rational.

Exr: If f_k is rational (of the same degree) and $f_k \to f$ on a positive measure set, then f is essentially rational.