KV-Topics
3/27/25

Higher Rank Oseledets (Oseledec):

Assume $\alpha : (\mathbb{R}^k \text{ or } \mathbb{Z}^k) \curvearrowright (X, \mu)$
ergodic and measure-preserving.

$\mathcal{A} : (\mathbb{R}^k \text{ or } \mathbb{Z}^k) \times X \to GL(d, \mathbb{R})$
is a cocycle

$$\mathcal{A}(a+b, x) = \mathcal{A}(a, \alpha(b)x)\, \mathcal{A}(b, x)$$

Assume $\mathcal{A}$ is mble,
by $|\mathcal{A}| \in L^k(\mu)$ $\left[L^{k, 1}_{(\mu)} \right]$.

Then $\exists\, \Lambda_0 \subseteq X$ such that such that $\mu(\Lambda_0) = 1$, a finite set $\Delta \subseteq (\mathbb{R}^k)^*$ and a μ-measurable splitting

$$\mathbb{R}^d = \bigoplus_{\lambda \in \Delta} E^{\lambda}_{(x)}, \quad x \in \Lambda_0$$

such that

a) $\mathcal{A}(q,x) E^{\lambda}_{(x)} = E^{\lambda}_{(qx)}$

b) $$\lim_{|q| \to \infty} \frac{|\log \mathcal{A}(q,x) v| - \lambda(q)}{|q|} = 0$$

whenever $v \in E^{\lambda}_{(x)}$

Furthermore

$$\lim_{|q| \to \infty} \frac{1}{|q|} \log \sin \measuredangle \left(E^{\lambda}_{(x)}, \bigoplus_{\mu \neq \lambda} E^{\mu}_{(x)} \right)$$

$\mathfrak{a}$ x.

Recall:

$$\begin{tikzcd} G \curvearrowright P \curvearrowleft H \subseteq GL(d,\mathbb{R}) \arrow[d] \\ G \curvearrowright (X,\mu) \end{tikzcd}$$

Let $\sigma : X \to P$ be a mble section of $P \to X$.

Define $\mathcal{A}(g,x)$ by

$$g \cdot \sigma(x) = \sigma(gx)\, \mathcal{A}(g,x)$$

- $\mathcal{A}(g,x)$ is unique since the H-action is free.

- Furthermore,

$$g_1 g_2 \sigma(x) = \sigma(g_1 g_2 x)\, \mathcal{A}(g_1 g_2, x)$$
$$= g_1\, \sigma(g_2 x)\, \mathcal{A}(g_2, x)$$
$$\hookrightarrow \quad = \sigma(g_1 g_2 x)\, \mathcal{A}(g_1, g_2 x)\, \mathcal{A}(g_2, x).$$

$$\boxed{\mathcal{A}(g_1 g_2, x) = \mathcal{A}(g_1, g_2 x)\, \mathcal{A}(g_2, x)}$$

"$\mathcal{A}$ = the defect of G-equivariance for σ"

- Apply Oseledets for the restriction to an $\mathbb{R}$-split Cartan $A \subseteq G$.

$\Rightarrow \exists\, \omega_0 : X \to Gr(k_1, d) \times Gr(k_2, d) \times \cdots \times Gr(k_p, d)$

where p = number of exponents,

$k_i = \dim(E^{\lambda_i})$.

Define $\omega : P \to \prod Gr(k_i, d)$ by writing $p = \sigma \circ \pi(p) \cdot \eta(p)$ and

$$\omega(p) = (\eta(p))^{-1} \cdot \omega_0 \circ \pi(p).$$

<u>Claim</u>: ω is H-equivariant and A-invariant.

<u>Note</u>: $ph = \sigma \circ \pi(p)\, \eta(p) h$

$$\Rightarrow \eta(ph) = \eta(p)$$

$$\omega(ph) = \eta(ph)^{-1}\, \omega_0 \circ \pi(p)$$

$$= h^{-1}\, \eta(p)^{-1}\, \omega_0 \circ \pi(p)$$

$$= h^{-1}\, \omega(p).$$

$(x = \pi(p))$

<u>A-invariance</u>: $\omega(ap) = \eta(ap)^{-1}\omega_0(ax)$

$$= \eta(ap)^{-1}\mathcal{A}(a,x)\,\omega_0(x)$$

$$\boxed{\begin{aligned} ap &= \sigma(ax)\,\eta(ap) \\ &= a\,\sigma(x)\,\mathcal{A}(a,x)^{-1}\eta(ap) \\ \eta(p) &= \mathcal{A}(a,x)^{-1}\eta(ap) \\ \eta(ap) &= \mathcal{A}(a,x)\,\eta(p). \end{aligned}}$$

$$= \eta(p)^{-1}\mathcal{A}(a,x)^{-1}\mathcal{A}(a,x)\,\omega_0(x)$$

$$= \eta(p)^{-1}\omega_0(x) = \omega(p).$$

Oseledets for Bundles:

Assume $A \curvearrowright P$ is an action by principle bundle automorphisms such that $B_{10}(0) \subseteq A$ has bounded orbit $\forall p \in P$. Then if $\rho : H \to GL(d, \mathbb{R})$ is a rep, $\exists$ A-inv. H-equivariant function $\omega : P \to \prod Gr(k_i, d)$ such that

if $v \in \omega(p)_i$, then $\subseteq B_p = (P \times \mathbb{R}^d) / H^\Delta$

$$\lim_{|a| \to \infty} \frac{\log |av| - \lambda(a)}{|a|} = 0.$$

(Breuer)

<u>Want to apply Zimmer superrigidity:</u>

$\forall p, \exists \eta_p : G \to N\backslash H$ ~~such that~~

and a section $\sigma : X \to P/N$

such that $\sigma(gx) = \sigma(x)\,\eta(g)$.

<u>What is N?</u>

N = pointwise stabilizer of
$\{\omega(gp) \mid g \in G\}$

= transformations which are Gr-block diagonal.

- For now, if $H = N \times H_0$, all growth rates (LEs) will be captured in H_0.

~~[illegible]~~ i.e. the Oseledets for the N-contribution has only 0 for an exponent.

- <u>Moral</u>: Property (T) + 0 ~~[illegible]~~ exponents $\Longrightarrow$ isometric

"Compact error"!

(This works in noble but fails in continuous category.)