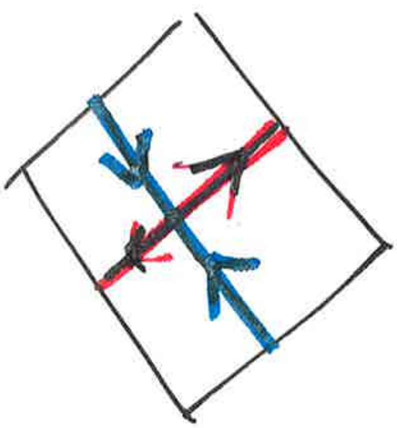
KV-Topics
3/25/25

Today: Things that can go wrong with Oseledets in Rank 1:

Katok Map (and Related Constructions)

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \circlearrowright \mathbb{T}^2$$

localize at $0 \in \mathbb{T}^2$

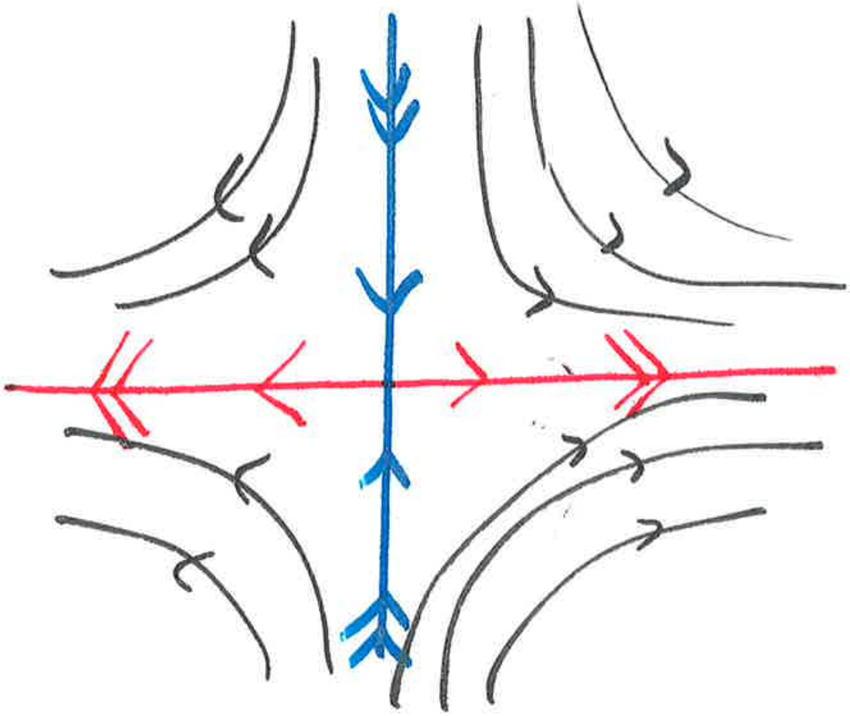


In suitable coordinates, near 0,

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e^{\chi} x \\ e^{-\chi} y \end{pmatrix}$$

Then A is the time-1 map of the vector field

$$\hookrightarrow \quad V_0 = \chi \cdot \left(x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} \right)$$



This flow preserves the foliation $xy = c$.

$$T(xy=c) = \ker\left(d(xy)\right)$$

$$= \ker\left(y\,dx + x\,dy\right)$$

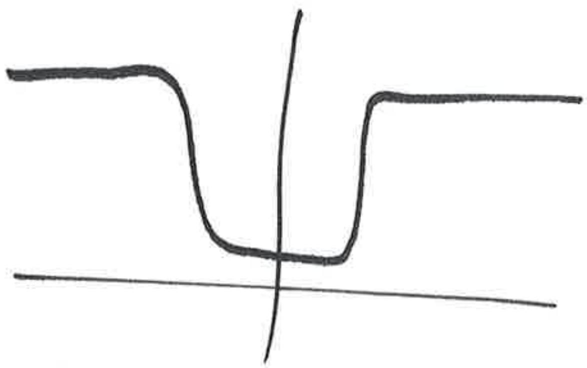
$$V_0 \in \ker \Rightarrow \mathbb{R} V_0 = T(xy=c).$$

ω

(Hu–Jiang–Jiang Construction)

Find $\eta : (-\varepsilon, \varepsilon) \to \mathbb{R}$ such that

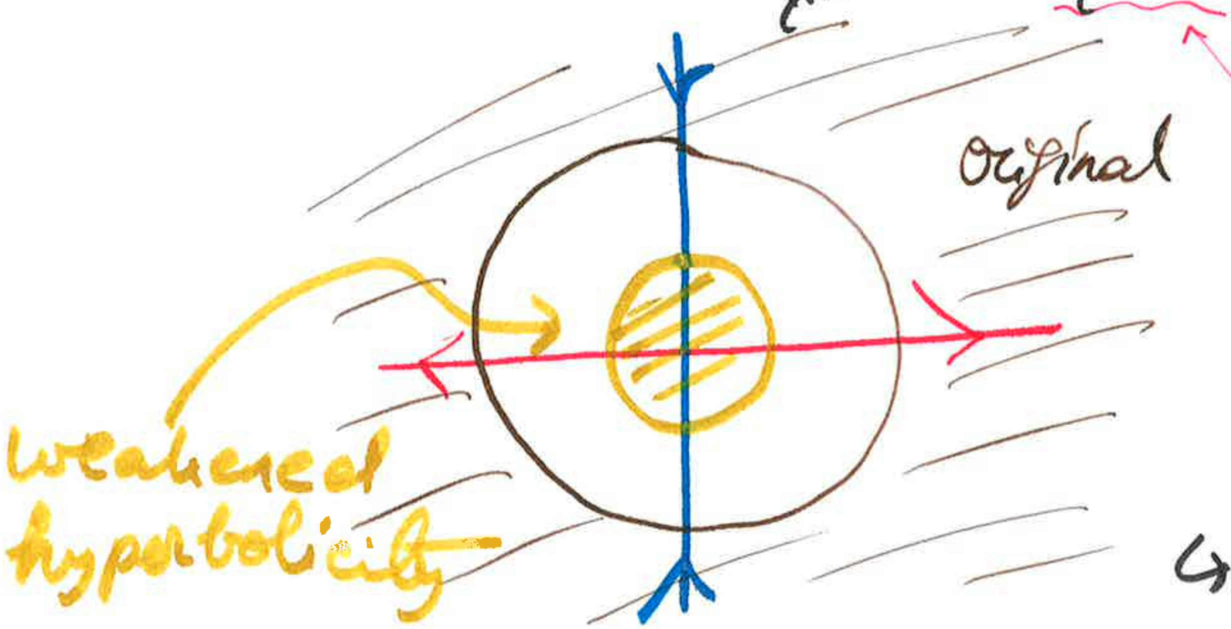
- $\eta \in C^\infty$
- $\eta(x) = \eta(-x)$
- $\eta\left(\left[\frac{2\varepsilon}{3}, 1\right)\right) \equiv 1$
- $\eta\left(\left[-\frac{\varepsilon}{3}, \frac{\varepsilon}{3}\right]\right) \equiv \delta << 1.$



(There is a slight issue with this possibility.)

Define $V(x,y) = \eta(xy)\, V_0(x,y)$

$\eta(x^2+y^2)$



Claim: Exponents w/r/t δ_0 and vol are both drastically diminished.

Since V is a multiple of V_0, it shares the same orbits.

Is this vol-preserving?

$$\omega = f\, dx \wedge dy$$

$$\mathcal{L}_V \omega = 0 ?$$

$$(d\iota_V + \iota_V d)\omega = 0$$

$d\omega \equiv 0$ since ω is a top form.

$$d(f \cdot \eta \cdot \iota_{V_0}\omega) = 0$$

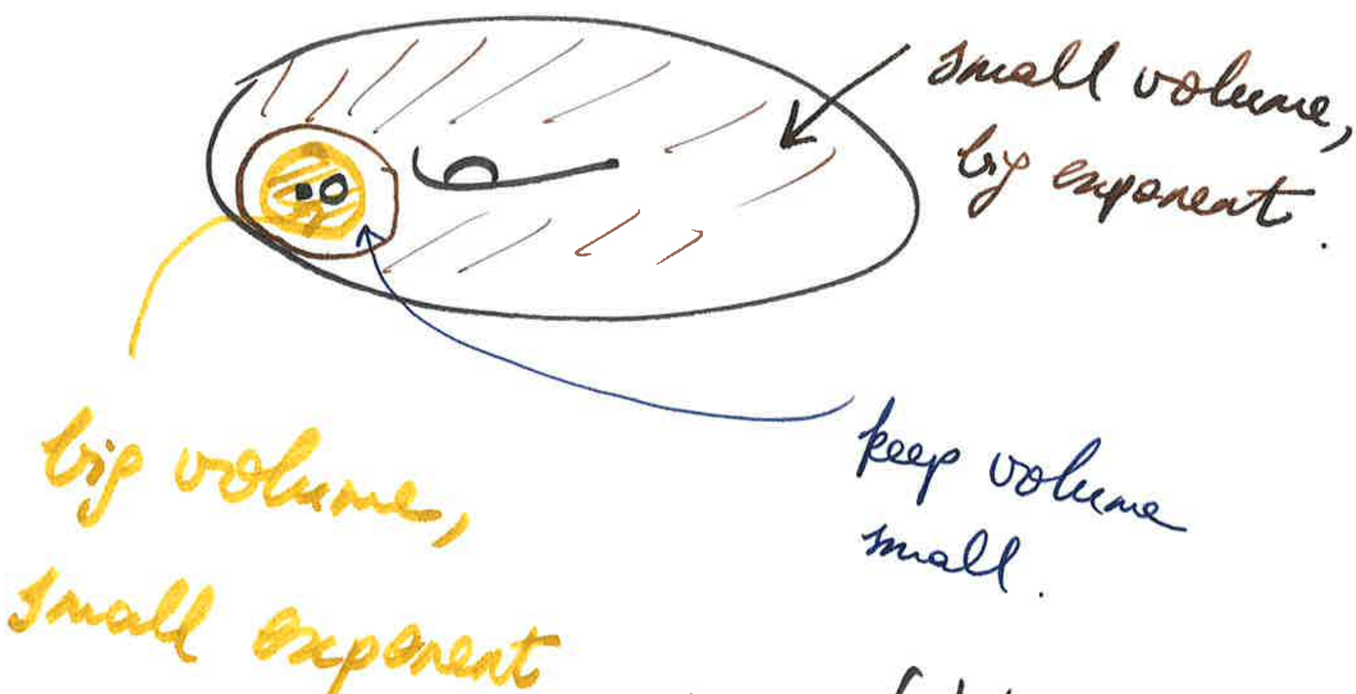
$$d(f \cdot \eta \cdot (x\,dy + y\,dx)) = 0$$

$$\Leftrightarrow d(f\eta)(x\,dy + y\,dx) = 0$$

$\Rightarrow$ $f \equiv \dfrac{1}{\eta}$ defines an invariant volume.

$$\Rightarrow \omega = \frac{1}{\eta \cdot \int \frac{1}{\eta} dx \wedge dy} dx \wedge dy$$

is an invariant smooth probability



$$LE \leq \int |df| \, \omega$$

$$\leq \boxed{\int \mathrm{I}} + \boxed{\int \mathrm{II}} + \boxed{\int \mathrm{III}}$$

$\hookrightarrow$

$\boxed{\int \mathrm{I}}$

$\boxed{\int \mathrm{II}}$

$\boxed{\int \mathrm{III}}$

• How to break even more? (KATOK)

Pick the function η extremely carefully.

LE @ 0 is 0

$\Rightarrow$ Not Anosov

— Destroys the Hölder regularity of E^s/E^u

SRB = equilibrium states for the geometric potential.

— Destroys Markov partitions

— $\exists$ 2 SRB measures: δ_0 & vol.

Katok-Lewis Blowup:

Introduce new coordinates on $B(0, \varepsilon)$.

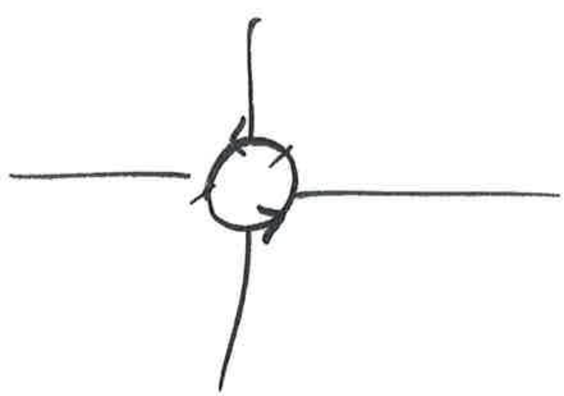
$$V = \mathbb{R}^2 \times \mathbb{R}P^1 \ni \Big((x_1, x_2), \underbrace{[v_1 : v_2]}_{\text{homog. coordinates.}} \Big)$$

$$\Rightarrow \quad x_1 v_2 - x_2 v_1 = 0$$

determines a subvariety $\mathcal{B}$.

"blowup of $\mathbb{R}^2$ at 0".

(nonorientable)



↳

$\Rightarrow \mathcal{B}$ is a <u>smooth</u> variety

(no singular points)

$(x, \theta) \in \mathbb{R}^2 \times S^1$

$$x_1 \sin(\theta) - x_2 \cos(\theta) = 0.$$

Check submersion:

$$d\left(x_1 \sin(\theta) - x_2 \cos(\theta)\right)$$

$$= \underbrace{\sin(\theta)\, dx_1 - \cos(\theta)\, dx_2}_{\neq 0}$$

$$+ \left(x_1 \cos(\theta) + x_2 \sin(\theta)\right) d\theta$$

$\hookrightarrow$

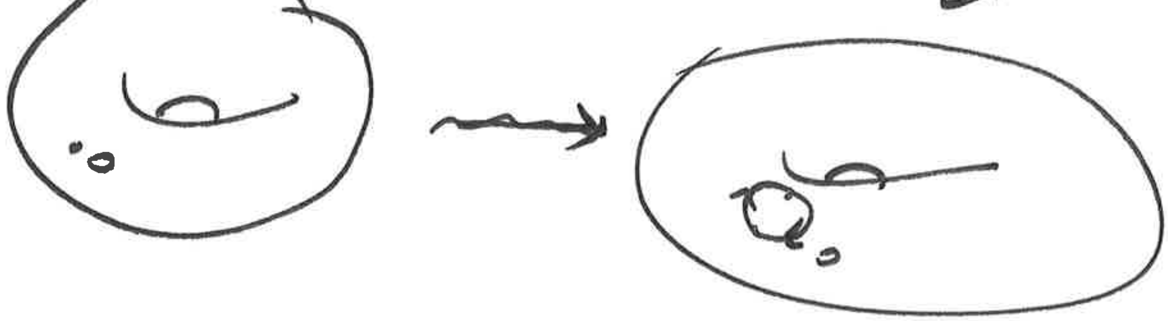
$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

$$A \circlearrowright V, \quad A(x,[v]) = (Ax,[Av])$$

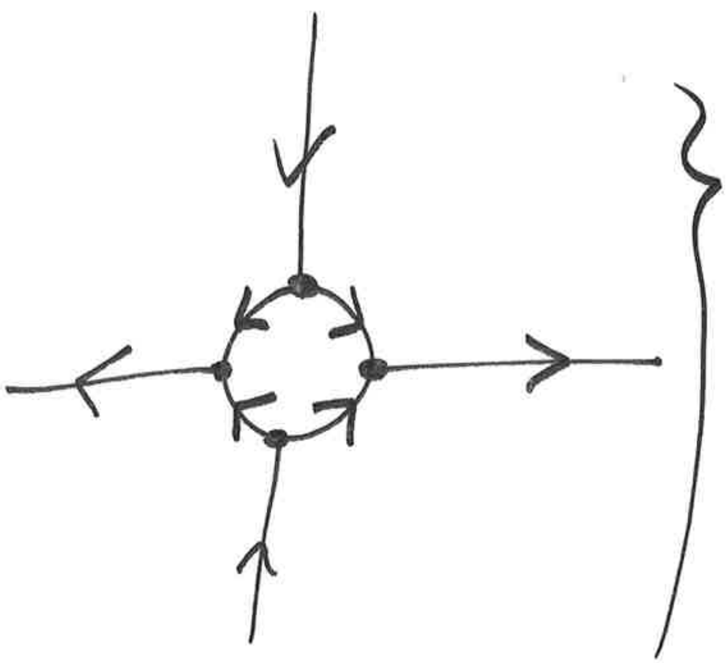
A preserves B, by linearity

$A \circlearrowright$

$A \circlearrowright$.



<u>Picture</u>:



Not Anosov.

Interesting Feature:

$\exists$ invariant volume which degenerates at the singular set, and

$$h_\omega(F) = h_{top}(F).$$

(Upcoming theorem by Brown–Lee.)

Open Q: Is this possible on $\mathbb{T}^2$?

Observe: If ~~Γ~~ $\Gamma \hookrightarrow SL(n, \mathbb{Z}) \curvearrowright \mathbb{T}^n$

$$\Gamma \curvearrowright V = \mathbb{R}^n \times \mathbb{R}P^{n-1}$$

$$\Gamma \curvearrowright B = \{(x, [v]) \mid x \in [v]\}$$

- This construction passes to Γ-actions.
- The associated objects are still only C^0 ~~able~~ and nonuniform. $\hookrightarrow$

- Conjecturally, $\exists$ open dense sets which have homogeneous structure, and one can inductively look at the singular sets

(conjecturally, manifolds with Γ-actions)

(When $\mathrm{rank}(G) \geq 2$).

<u>Katok-RH</u>: This holds under maximal rank Cartan assumptions, without full understanding of singularity.