KV – Topics
3/18/25

Using Rationality: Tameness of Actions:

Recall: $H \curvearrowright X$ is tame if it is countably separated (equivalently, every orbit is locally closed).

(We used tameness before:
$H \curvearrowright V$, $\mathcal{F} = \mathrm{Fun}(Y, V)$
$H \curvearrowright \mathcal{F}$: $[h \cdot f](y) = h \cdot f(y)$.
).

Ex: Let $\mathcal{F}(\mathbb{R};\mathbb{R})$ be a function space.

(e.g. $\mathcal{F} = \underbrace{C^0, C^1, C^\infty, \text{Analytic}}_{C^0 \text{ compact-open}}$ (?), Meas, ...)
finite-open

$\mathbb{R}\times\mathbb{R} \curvearrowright \mathcal{F}$, $[(t,s)\cdot f](x) = f(x-t)+s$

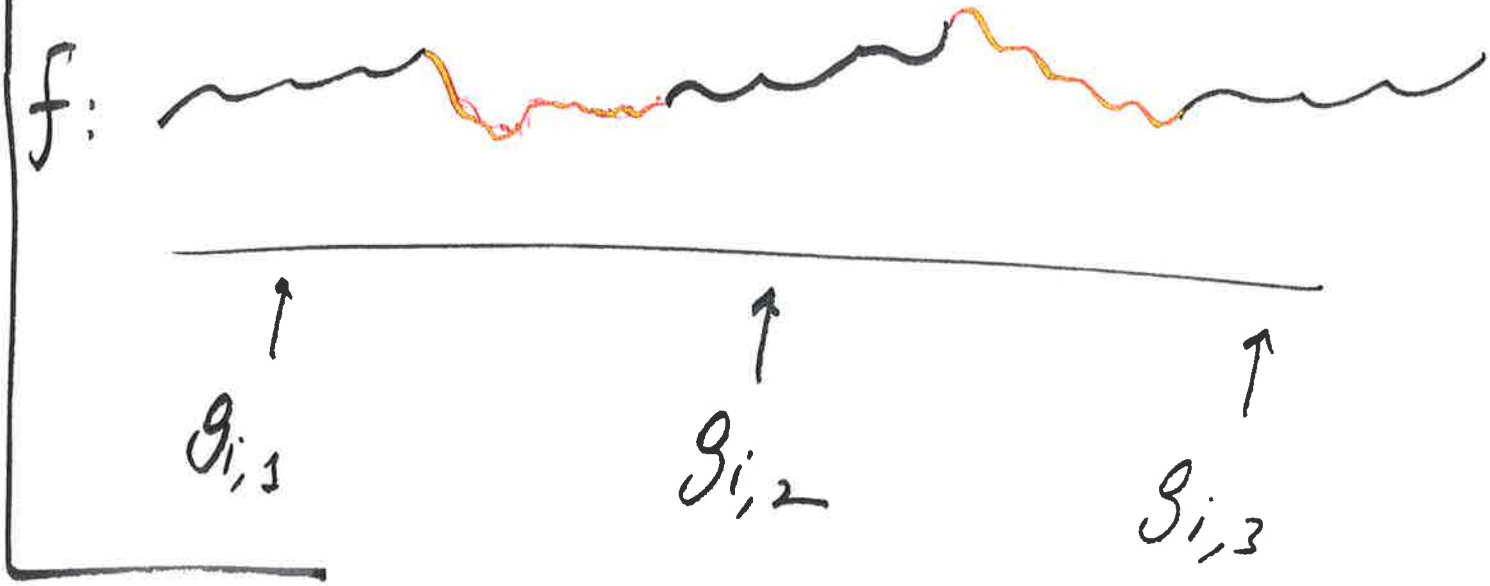
$\{0\}\times\mathbb{R}$-action is tame.

$\mathbb{R}\times\{0\}$-action is <u>not</u> tame (except perhaps for C^ω).

Claim: $\exists f \in \mathcal{F}$ such that the $\mathbb{R}\times\{0\}$ orbit of f is dense.

Consider $\mathcal{F}([-n,n];\mathbb{R})$, which is separable

$\Rightarrow \exists \{g_{i,n}\}_i$ such that $\overline{\{g_{i,n}\}_i} = \mathcal{F}([-n,n];\mathbb{R})$



<u>Theorem</u>: Let G be an $\mathbb{R}$-algebraic group, and V be a variety. Assume $H \curvearrowright V$ is rational, and set $\mathcal{R} = \mathrm{Rat}(G,V)$. Define

$$G \times H \curvearrowright \mathcal{R} \text{ by}$$

$$\big[(g,h)\cdot f\big](x) = h \cdot f(gv)$$

Then the $G \times H$ action is tame.

Ex: If $f(x) = x^2$,

$$[(t,s) \cdot f](x) = (x-t)^2 + s$$
$$= x^2 - 2tx + (t^2 - s)$$

$\Rightarrow \mathbb{R}^2$-orbit is closed.

Pf idea: Degree is preserved, and the action is rational in coefficient space.

Apply Rosenlicht.

Trying to apply this:

Recall: We're assuming

$$\begin{tikzcd} G \curvearrowright P \curvearrowleft H, \arrow[d] & \\ G \curvearrowright (X,\mu) & \end{tikzcd}$$

and $\exists\, \omega : P \to H$ H-equivariant

$$\boxed{\begin{array}{l} P \curvearrowleft H : \text{alg. hull} \\ V = Gr(\cdot) \end{array}}$$

$\Leftrightarrow$ $\boxed{\exists \text{ section of associated bundle } V \to B \downarrow X}$

Define $\Phi : P \longrightarrow \mathrm{Fun}(G, V)$

$$p \longmapsto \left[g \longmapsto \omega \text{ [redacted] } (gp) \right].$$

We developed the tools to show that Φ takes values in $\mathrm{Rat}(G, V)$.

<u>Step 1</u>: ω [redacted] is rational along root subgroups. ← (This step needs higher rank.)

<u>Step 2</u>: (Margulis trick)

Φ is rational along Borel subgroups

<u>Step 3</u>: Bruhat decomposition

$$G = \bigsqcup_{w \in W} BwB, \quad W = \text{Weyl group}.$$

<u>Note</u> : Φ is not A-invariant

$$\Phi(ap)(g) = \omega(gap).$$

$$\neq \omega(agp) = \omega(gp).$$

Φ descends to a function

$$\overline{\Phi} : X \to \mathrm{Rat}(G;V)/H$$

by H-equivariance.

Quotient further on the right

to get $\check{\Phi} : X \to \mathrm{Rat}(G;V)/G\times H$.

<u>Claim</u> : $\check{\Phi}$ is G-invariant

Pf. of Claim:

$$\check{\Phi}(gx) = (G\times H)\cdot \Phi(gp)$$

(for p covering x.)

$$= (G\times H)\cdot\left(\Phi(p)\circ L_g\right)$$

$$= \left((Gg)\times H\right)\cdot \Phi(p)$$

$$= \check{\Phi}(x).$$

$\Rightarrow$ $\check{\Phi}$ is constant by ergodicity/transitivity.

$\Rightarrow$ $\Phi(p) \subseteq (G\times H)\cdot f_0$, $f_0 \in Rat(G,V)$

image of Φ

Given $p, q \in P$, $\exists\, (g,h) \in G \times H$ such that $\Phi(p) = h \cdot \Phi(gq)$

$\Rightarrow \forall g \in G, \forall x \cdot p \in P,$

$\exists\, \tilde{\eta}_p(g) \in H$ such that

$$\boxed{\Phi(gp) = \tilde{\eta}_p(g) \cdot \Phi(p)}$$

<u>Q</u> : Is $\tilde{\eta}_p$ a homomorphism?

<u>Issue</u> : $\tilde{\eta}_p$ is not uniquely defined.

$\Rightarrow$ We have to do the dance of considering $Stab(\Phi(p)) \subseteq H$.

$\hookrightarrow$

$$S_p = \mathrm{Stab}(\Phi(p)) = \bigcap_{g \in G} \mathrm{Stab}(\omega(gp))$$

$$\ldots \quad \tilde{\eta}_p(g) \in \mathrm{Normalizer}(S_p)$$

Assuming $\tilde{\eta}_p$ well-defined,

$$\tilde{\eta}_p(g_1 g_2)\, \Phi(p) = \Phi(g_1 g_2 p)$$

$$= \tilde{\eta}_{g_2 p}(g_1)\, \Phi(g_2 p)$$

$$= \tilde{\eta}_{g_2 p}(g_1) \cdot \tilde{\eta}_p(g_2)\, \Phi(p)$$

Last step: $\tilde{\eta}_{gp} = \tilde{\eta}_p$ (following the G_α - proof).

Upshot: $\exists$ family of homomorphisms

$$\eta_p : G \to H/K \quad \text{such that}$$

$$\omega(gp) = \omega(p)\,\eta_p(g)^{-1} \quad \text{and}$$

$$K = \ker(H \curvearrowright \omega p).$$

Look ahead: $V = Gr(\,.\,)$

$H \curvearrowright V$ linear action

ω assigns the stable manifold

$K = \{$transformations preserving every stable distribution$\}$

= compact, since it has 0 exponents

property (T) $\Longrightarrow$ compact

Final Note: Bruhat decomposition

for $SL(2,\mathbb{R})$

$$G = \bigsqcup_{w \in W} BwB.$$

$$= \underbrace{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}}_{w = e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} \sqcup \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}}_{w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}$$

$$B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

"big cell"

- The big cell always has w to the "longest" Weyl group element.
- For $SL(d,\mathbb{R})$, $W = S_d$ so the longest w is $(1\ 2 \ldots d)$.