KV-Topics
1/9/25

Dynamical Interpretation of the Adjoint Representation Ad:

G: Lie group
$\Gamma \leq G$ lattice

Put a metric on G/Γ?

Choose a right invariant metric on G and project it to G/Γ.

Each $g \in G$ induces a dynamical system on $G/\Gamma = \{g\Gamma \mid g \in G\}$ by left multiplication.

$$L_g : G/\Gamma \to G/\Gamma, \quad x\Gamma \mapsto gx\Gamma$$

What does the derivative transformation dL_g look like?

$$dL_g : T_h G \longrightarrow T_{gh} G$$

the same space.

$$\phi : T(G/\Gamma) \to G/\Gamma \times T_e G$$

$$(v, g\Gamma) \mapsto (g\Gamma, dR_{g^{-1}}(v))$$

$v \in T_{g\Gamma}(G/\Gamma)$

<u>HW</u>: Check that this is a well defined vector bundle isomorphism.

$$dL_g(v, h\Gamma) = (dL_g(v), gh\Gamma)$$

trivialization

$$dR_{h^{-1}}(v)$$

$$dR_{(gh)^{-1}}\, dL_g(v)$$

$$dR_{h^{-1}}\, dR_{g^{-1}}\, dL_g(v)$$

$$Ad(g) = dR_{g^{-1}}\, dL_g$$

$$\boxed{\text{Thus the derivative is determined by } Ad(g).}$$

Suspension Ex from Last Time.

$$\mathbb{Z} \subseteq \mathbb{R}, \qquad A \in SL(2, \mathbb{Z})$$

$$A = \exp(X)$$

$$\left(\mathbb{R} \ltimes_X \mathbb{R}^2\right) \Big/ \left(\mathbb{Z} \ltimes_A \mathbb{Z}^2\right).$$

HW: Show that for $H = \mathbb{R} \ltimes_X \mathbb{R}^2$,

$$H \cong \left\{ \left(\begin{array}{c|c} A^t & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline v & 1 \end{array} \right) \;\middle|\; \begin{matrix} t \in \mathbb{R} \\ v \in \mathbb{R}^2 \end{matrix} \right\} \quad \text{and}$$

$$\mathcal{L}ie(H) = \text{Lie algebra of } H = \left\{ \left(\begin{array}{c|c} tX & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline v & 0 \end{array} \right) \right\}$$

$$\mathrm{Ad}(A)(X) = 0$$

$$\mathrm{Ad}(A)(v) = Av$$

Solvable group?

H is called solvable if

$$\left(\begin{array}{l} H_0 = H \\ H_1 = [H_0, H_0] \\ H_{n+1} = [H_n, H_n] \end{array}\right) \quad H_0 \to H_1 \to H_2 \to H_3 \to e$$

$$(H_n = H_n^{solv})$$

HW: The above group H is solvable.

Nilpotent group?

H is called nilpotent if

$$\left(\begin{array}{l} H^0 = H \\ H^1 = [H, H^0] \\ H^{n+1} = [H, H^n] \end{array}\right) \quad \cancel{H_0 \to H_1}$$

$$H^0 \to H^1 \to H^2 \to H^3 \to e.$$

$$(H^n = H_n^{nil}).$$

<u>HW</u>: Show that H is nilpotent

$\iff$ the only eigenvalue of X is 0.

(e.g. Heisenberg).

<u>Ex</u>: $H = \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix}$ upper triangular matrices are solvable.

$$[\mathrm{Lie}(H), \mathrm{Lie}(H)] = \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix} \mathrm{Lie}(H_1^{solv})$$

$$\mathrm{Lie}(H_2^{solv}) = \begin{pmatrix} 0 & 0 & & * \\ 0 & \ddots & \ddots & \\ & \ddots & \ddots & 0 \\ 0 & & 0 & 0 \end{pmatrix}$$

$\vdots$

Ex: $H = \begin{pmatrix} * & \cdots & * \\ 0 & \ddots & \vdots \\ 0 & \cdots\ 0 & * \end{pmatrix}$ upper triangular matrices are not nilpotent.

Lie (H_1^{nil}) = Lie (H_2^{nil}) . . .

* Solvability and nilpotence are hereditary.

Ex: $G = \begin{pmatrix} 1 & * & \cdots & * \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & 1 \end{pmatrix}$ is nilpotent.

HW: Prove this
(useful fact: $[E_{ij}, E_{jk}] = E_{ik}$)

Thm (Lie): Every solvable Lie group is isomorphic to a closed subgroup of $H_{\mathbb{C}}$ = upper triangulars with complex entries.

s.c.

Thm (Engel): Every nilpotent Lie group is isomorphic to a closed subgroups of G

(more on Lie & Engel thms later).

Killing Form: (following Helgason).

Let $\mathfrak{g}$ be a Lie algebra.

$$X \in \mathfrak{g} \longrightarrow \mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g} \quad \text{"little ad"}$$
$$Y \mapsto [X, Y].$$

Def: The Killing form on $\mathfrak{g}$ is a bilinear form defined by

$$B(X, Y) = \mathrm{Tr}\left(\mathrm{ad}_X \, \mathrm{ad}_Y\right).$$

HW:
- B is bilinear
- B is symmetric
- $\forall \sigma \in \mathrm{Aut}(\mathfrak{g}) : B(\sigma X, \sigma Y) = B(X, Y)$

HW: Find the Killing forms for ...

- $\mathbb{R}^3$
- $\mathbb{R} \ltimes_X \mathbb{R}^2$
- ~~SO(3)~~ $so(3)$
- $sl(2, \mathbb{R})$

- $\mathfrak{h} \subseteq \mathfrak{g}$ is an ideal if

$$[\mathfrak{g}, \mathfrak{h}] \subseteq \mathfrak{h}.$$

Lemma: If $\mathfrak{h}$ is an ideal in $\mathfrak{g}$, then $B_{\mathfrak{g}}\big|_{\mathfrak{h}} = B_{\mathfrak{h}}$.

Pf: Pick a basis $\underbrace{X_1, \dots, X_n}_{\mathfrak{h}}, \underbrace{X_{n+1}, \dots, X_m}_{\text{transverse to } \mathfrak{h}}$ of $\mathfrak{g}$

$$B_{\mathfrak{g}}(X_i, X_j) = tr\left(ad_{X_i}\, ad_{X_j}\right)$$

$$= \underbrace{\sum_{k=1}^{n} X_k^*\left([X_i, [X_j, X_k]]\right)}_{= B_{\mathfrak{h}}(X_i, X_j)}$$

$$+ \sum_{k=n+1}^{m} \text{———} \,\text{''}\, \text{———}$$

vanishes for $X_i, X_j \in \mathfrak{h}$

Def: $\mathfrak{g}$ is called semisimple if $B_{\mathfrak{g}}$ is a nondegenerate bilinear form.

FACT: Every simple Lie algebra $(\neq \mathbb{R})$ is semisimple.

Lemma: $\mathfrak{g}$ is compact and center-free iff B is negative definite.

Pf: $(\Leftarrow)$ B is negative def

$\Leftrightarrow$ $-B$ is a metric

$\Rightarrow$ $O(B) = \left\{ \begin{array}{c} \text{transformations} \\ \text{preserving } B \end{array} \right\}$

is a compact group.

$\boxed{\begin{array}{l} \mathfrak{g} \text{ is } \underline{\text{compact}} \\ \text{if } \{\exp(\mathrm{ad}_X) \mid X \in \mathfrak{g}\} \\ \text{is compact.} \end{array}}$

$$\langle \exp(\mathrm{ad}_X) \mid X \in \mathfrak{g} \rangle \underset{\text{closed}}{\subseteq} O(B)$$

$(\Rightarrow)$ (harder direction)

<u>Case</u>: $\mathrm{Ad}(\mathfrak{g}) \subseteq SO(Q)$

(adjoint group of $\mathfrak{g}$) for some quadratic form Q.

$\Rightarrow$ $\mathfrak{g} \subseteq \mathfrak{so}(Q)$

$\Rightarrow$ Every ad_X is skew-symmetric w/r/t $X_1, X_2, \ldots, X_n$, a basis.

$$B(X,X) = \mathrm{tr}\,(\mathrm{ad}_X\, \mathrm{ad}_X) \qquad \begin{pmatrix} a_{ij} : \text{entries} \\ \text{of } \mathrm{ad}_X . \end{pmatrix}$$

$$= \sum_{i,j=1}^{n} a_{ij} a_{ji} = \sum_{i,j=1}^{n} - a_{ij}^2 < 0$$

(by the center-free assumption.).

Def: $X \in \mathfrak{g}$ is called F-semisimple ($F=\mathbb{R}$ or $F=\mathbb{C}$) if ad_X is diagonalizable over F.

$\mathfrak{h} \subseteq \mathfrak{g}$ is an (<u>F</u>-split) <u>Cartan subalgebra</u> if it is ~~abe~~ abelian,

every $X \in \mathfrak{h}$ is F-semisimple,

and $\mathfrak{h}$ is maximal among such subalgebras.

Ex: $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$

$$\mathfrak{a} \subseteq \mathfrak{g}, \quad \mathfrak{a} = \left\{ \begin{pmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_n \end{pmatrix} \;\middle|\; \begin{array}{l} t_i \in \mathbb{R} \\ \sum t_i = 0 \end{array} \right\}$$

is an $\mathbb{R}$-split Cartan subalgebra.

$$\mathfrak{h} \subseteq \mathfrak{g}, \quad \mathfrak{h} = \left\{ \begin{pmatrix} z_1 & & 0 \\ & \ddots & \\ 0 & & z_n \end{pmatrix} \;\middle|\; \begin{array}{l} z_i \in \mathbb{C} \\ \sum z_i = 0 \end{array} \right\}$$

is a $\mathbb{C}$-split Cartan subalgebra.

$$X = \operatorname{diag}(t_1, \dots, t_n) \in \mathfrak{sl}(n, \mathbb{C})$$

$$\operatorname{ad}_X (E_{ij}) = \underbrace{(t_i - t_j)}_{r_{ij}(X)} E_{ij}$$

$r_{ij}(X)$ ← important function (Lyapunov exponent)

$$\operatorname{Ad}(\exp(tX)) = \exp(t \operatorname{ad}_X).$$

preview :

A : $\mathbb{R}$-split Cartan.

$$Z(A) = \underset{\substack{\uparrow \\ \text{compact}}}{M} \times A$$

has the most amount of hyperbolicity.

$$A \curvearrowright M\backslash G/\Gamma$$