KV- Topics

1/~~7~~/25

<u>Def</u>: Let G be a Lie group. A (right) Haar form on G is a top form on G which satisfies

$$(R_g)^* \omega = \omega \qquad R_g : G \to G, \quad h \mapsto hg$$

[i.e. ω is right-invariant]

<u>Lemma / HW</u>: Let G be a Lie group.

- If ω_1, ω_2 are two nonzero Haar forms, $\exists \lambda \in \mathbb{R} : \omega_1 = \lambda \omega_2$.
- If ω is a Haar form, so is

$$(L_g)^* \omega.$$

Let $\lambda : G \to \mathbb{R}$ be defined by

$$\lambda(g) = \frac{(L_{g^{-1}})^* \omega}{\omega},$$ where ω is a Haar form.

<u>Claim</u>: λ is a homomorphism

G is called <u>unimodular</u> if λ is trivial.

Let $\Gamma \subseteq G$ be a discrete subgroup.
Then $G/\Gamma = \{g\Gamma \mid g \in G\}$ is a well-defined manifold.

The <u>right</u> Haar form descends to G/Γ.

All discrete quotients will be on the right.

<u>Def</u>: Γ is a <u>lattice</u> if is discrete and $\int_{G/\Gamma} \omega < \infty$.

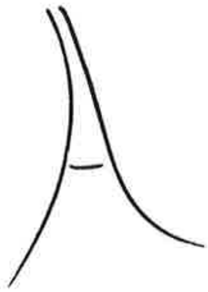
<u>Ex</u>: $\mathbb{Z} \subseteq \mathbb{R}$

$\mathbb{Z}^k \subseteq \mathbb{R}^k$

$SL(n,\mathbb{Z}) \subseteq SL(n,\mathbb{R})$ (Borel).

Def: Γ is a uniform lattice if G/Γ is compact.

Non-uniform lattices have cusps.



HW: If G is a lattice, then it is unimodular.

Think about:

Smooth actions of Lie groups and their lattices.

Ex: Automorphism actions

$$\Gamma = SL(n, \mathbb{Z}) \quad \alpha : \Gamma \curvearrowright \mathbb{T}^n$$

$$\left(\alpha : \Gamma \longrightarrow \mathrm{Diff}^{\infty}(\mathbb{T}^n) \right)$$

$$\alpha(\gamma)\left[v \,(\mathrm{mod}\ \mathbb{Z}^n) \right] = \gamma v \,(\mathrm{mod}\ \mathbb{Z}^n)$$

moral of the story:

$$\begin{array}{ccc} & & \mathrm{Aut}(\mathbb{T}^n) \\ & \nearrow & \cap \\ \Gamma & \xrightarrow{\alpha} & \mathrm{Diff}^{\infty}(\mathbb{T}^n) \end{array}$$

<u>Suspensions</u>: Let $\Gamma \subseteq G$ be a lattice and $\alpha : \Gamma \curvearrowright X$.

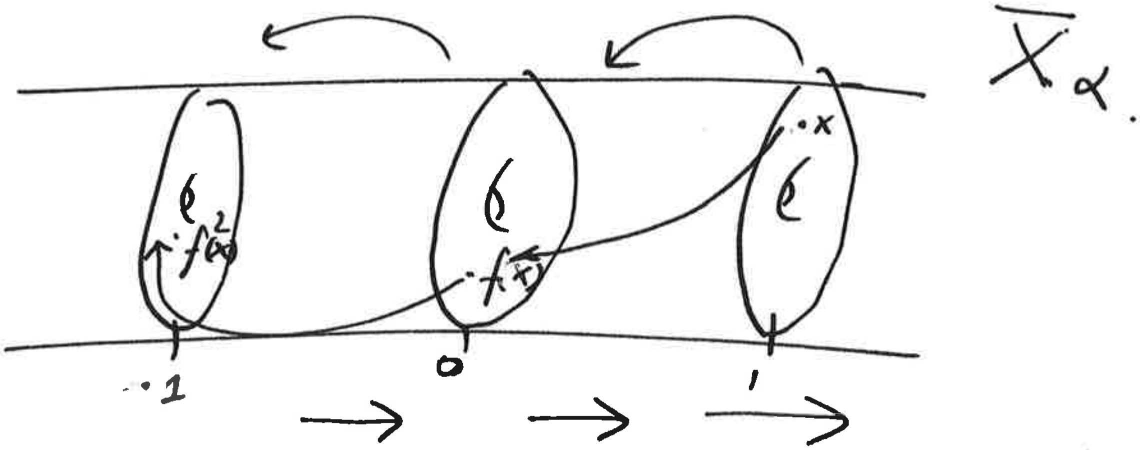
$\bar{X}_\alpha = G \times X$ has two actions on it:

$$\begin{cases} G \curvearrowright \bar{X}_\alpha & g \cdot (h, x) = (gh, x) \\ \bar{X}_\alpha \curvearrowleft \Gamma & (h, x) \cdot \gamma = (h\gamma, \alpha(\gamma^{-1}) x). \end{cases}$$

These two actions ~~comp~~ commute.

$X_\alpha = \bar{X}_\alpha / \Gamma$. $\quad G \curvearrowright X_\alpha \leftarrow$ the suspension of the Γ-action.

Ex: $\mathbb{Z} \subseteq \mathbb{R}$, $\alpha : \mathbb{Z} \curvearrowright X$, $\alpha = \langle f \rangle$.



Suspending Automorphism Actions:

$\Gamma \subseteq G$ is a lattice.

$\alpha : \Gamma \to SL(n, \mathbb{Z}) \subseteq \mathrm{Diff}(\mathbb{T}^n)$.

Assume α extends to a homomorphism $\tilde{\alpha} : G \to SL(n, \mathbb{R})$ such that $\tilde{\alpha}|_\Gamma = \alpha$.

Define $H = \underbrace{G \ltimes_{\tilde{\alpha}} \mathbb{R}^n}$

As a manifold: $G \times \mathbb{R}^n$

$$(g, v) \cdot (h, w) = (gh, \tilde{\alpha}(g) w + v)$$

Set $\Lambda = \Gamma \ltimes_{\alpha} \mathbb{Z}^n \subseteq H$.

<u>HW</u>: $G \curvearrowright H/\Lambda$ is C^∞ conjugated

$G \curvearrowright (\mathbb{T}^n)_\alpha$.

One of the goals for this semester:

<u>Theorem</u> (Margulis): Let G be a semisimple Lie group with $\mathbb{R}$-rank ≥ 2 (e.g. $SL(n, \mathbb{R})$, $n \geq 3$). If $\Gamma \leq G$ is a lattice, every homomorphism $\rho : \Gamma \to SL(n, \mathbb{Z})$ extends to a homomorphism $\tilde{\rho} : G \to SL(n, \mathbb{R})$

[slight lie].

This fails <u>wildly</u>! for $SL(2,\mathbb{R})$

$F_2 \subseteq SL(2,\mathbb{R})$ is a lattice.

$\uparrow$

free group on two elements.

<u>Conj</u> (<u>Zimmer</u>): If G is simple, $\mathbb{R}$-rank ≥ 2 and $\Gamma \underset{\text{lat}}{\subseteq} G$, then any $\Gamma \overset{C^\infty}{\curvearrowright} X$ has algebraic building blocks.

$\boxed{\text{same for } G\text{-actions.}}$

two prototypes: boundary actions, (projectivized)

actions by automorphisms.

More basic version of algebraic suspension:

$$f = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \curvearrowright \mathbb{T}^2$$

$$\mathbb{Z} = \langle f \rangle \subseteq \mathbb{R}.$$

$$\mathbb{Z} \to SL(2,\mathbb{Z})$$

$$\mathbb{R} \to SL(2,\mathbb{R})$$

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = B \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} B^{-1}$$

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \exp\left(B \begin{pmatrix} \log \lambda & 0 \\ 0 & -\log \lambda \end{pmatrix} B^{-1} \right).$$

$$\cancel{t} \quad t \mapsto \exp\left(t \, B \begin{pmatrix} \log \lambda & 0 \\ 0 & -\log \lambda \end{pmatrix} B^{-1} \right).$$

$$\mathbb{R} \ltimes \mathbb{R}^2$$

$$(t,v)\cdot(s,w) = (t+s, \rho(t)w + v)$$

$$(t,v)\cdot(s,0) = (t+s, v)$$

$$(t,v)\cdot(0,w) = (t, \rho(t)w + v)$$

$$(s,0)\cdot(t,v) = (t+s, \rho(s)v)$$

• All ~~r~~ right actions are structural, all left actions are dynamical.

<u>HW</u>: Show that $\mathbb{R} \ltimes \mathbb{R}^2$ is solvable but not nilpotent for our homomorphism ρ.

<u>Exercise</u>: Choose <u>any</u> $\rho : \mathbb{R} \to GL(2, \mathbb{R})$.
Find the right Haar form for $\mathbb{R} \ltimes_\rho \mathbb{R}^2 = H$.
Show that H is unimodular
$\iff \rho(\mathbb{R}) \subseteq SL(2, \mathbb{R})$.
