KV-Topics
1/30/25

<u>Recall</u>: Margulis superrigidity

$\mathrm{rank}_{\mathbb{R}}(G) \geq 2$, G simple, $\Gamma \leq G$ lattice

If $\rho : \Gamma \to GL(d, \mathbb{R})$ is a rep,
then $\exists \tilde{\rho} : G \to GL(d, \mathbb{R})$ such that

$$\tilde{\rho}|_{\Gamma} = \rho^{*}$$

($*$: slight lie).

<u>Today</u>: More on the lie.
(via restriction of scalars)

<u>Ex</u>: Consider a lattice from restriction of scalars

e.g. $F = \mathbb{Q}[\sqrt{2}] = \mathbb{Q}[\alpha]/(\alpha^2 - 2)$

$\mathcal{O} = \mathbb{Z}[\sqrt{2}]$

$\hookrightarrow$

$B : F^3 \times F^3 \longrightarrow F$

$B = x^2 + y^2 + \alpha z^2.$

<u>Note</u>: $F \cong \mathbb{Q}^2$ as a vector space with basis $\langle 1, \alpha \rangle$.

For multiplication,

$$1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \quad \alpha \mapsto \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}.$$

$$B(v,w) = v^T \left(\begin{array}{c|c|c} I_2 & 0 & 0 \\ \hline 0 & I_2 & 0 \\ \hline 0 & 0 & \begin{smallmatrix} 0 & 2 \\ 1 & 0 \end{smallmatrix} \end{array} \right) w$$

on $\mathbb{Q}^2 \cong F$.

$\Rightarrow$ $SO(B)$ is a $\mathbb{Q}$-group with these choices.

$\Rightarrow$ $SO(B;\mathcal{O})$ are the $\mathbb{Z}$-points.

$SO(B;\mathbb{R}) = SO(3;\mathbb{R}) \times SO(2,1;\mathbb{R})$

$\hookrightarrow$

$\Rightarrow$ $SO(B;\Theta)$ is the $\mathbb{Z}$-points of $SO(3)\times SO(2,1)$ in some embedding, but not just $SO(2,1)$

$\Rightarrow$ $\exists$ homomorphism $\rho : SO(B;\Theta) \to GL(6,\mathbb{R})$ which does not extend to a homomorphism $\hat{\rho} : SO(2,1) \to GL(6;\mathbb{R})$.

<u>Margulis superrigidity</u> (Again, Corrected)

$\operatorname{rank}_{\mathbb{R}}(G) \geq 2$, G simple, $\Gamma \leq G$ lattice. If $\rho : \Gamma \to GL(d,\mathbb{R})$ is a rep, then $\exists K$ compact, a diagonal embedding $i : \Gamma \to G \times K$ and a rep $\tilde{\rho} : G\times K \to GL(d,\mathbb{R})$ such that $\tilde{\rho}\circ i = \rho$.

Exr: Show that this is equivalent to:

$\exists \tilde{\rho}: G \to GL(d, \mathbb{R})$ and a rep

$k: \Gamma \to GL(d, \mathbb{R})$ such that

$[\tilde{\rho}(G), k(\Gamma)] = e$ and

$k(\Gamma)$ takes values in a compact group.

(the conclusion of the corrected Margulis superrigidity).

Relating to Alg. Hulls:

Setting: $\Gamma \subseteq G \times K$ lattice from restriction of scalars

$$\rho: \Gamma \to SL(d, \mathbb{Z})$$

Exr: Show the suspension action on $(G \times \mathbb{T}^d)/\sim$ $(g, x) \sim (g\gamma, \gamma^{-1}x)$ is conjugated to ... $(\mathbb{R}^d\ ?)$

$\hookrightarrow$

...

$$G \curvearrowright K \backslash \Big((G\times K) \ltimes_{\tilde\rho} \mathbb{R}^d \Big) \Big/ \Gamma \ltimes_\rho \mathbb{Z}^d$$

(Harder:) Furthermore, show that it is not conjugated to a homogeneous action.

Exr: In the same setting as the previous exr., show that the algebraic hull of the G-action is $G\times K$. (rmk: C^0/C^∞ alg. hulls are all the same for alg. actions.)

Theorem (Zimmer) (Zimmer cocycle superrigidity): Suppose $\mathrm{rank}_{\mathbb{R}}(G)\geq 2$, G simple, ~~...~~.

Let $G \curvearrowright (X,\mu)$ be ergodic & μ-preserving (μ probability), $G \curvearrowright P \curvearrowleft H$ be an action by bundle automorphisms of a principal H-bundle

$$\begin{array}{c} P \\ \downarrow \\ (X,\mu) \\ \circlearrowleft G \end{array}$$

Then the (measurable) algebraic hull Q with structure group L is semisimple.

Furthermore, $\exists K \triangleleft L$ compact, a homomorphism $\rho : G \to L/K$ and a measurable section $\omega : X \to P/K$ such that $g \cdot \omega(x) = \omega(gx) \cdot \rho(g)$

the proof that the alg. hull $Q \triangleleft L$ is semisimple uses Oseledets

In cocycle language, every cocycle is cohomologous to a homomorphism modulo compact noise.

Pf of Margulis superrigidity via Zimmer Cocycle superrigidity:

Build a suspension from $\rho : \Gamma \to GL(d,\mathbb{R})$

$$P = \left(G \times GL(d,\mathbb{R})\right)/\sim$$

$$(g,h) \sim (g\gamma, \rho(\gamma)^{-1}h)$$

P is a principal $GL(d,\mathbb{R})$-bundle over G/Γ, G acts by bundle automorphisms.

If L = structure group of the algebraic hull, L = semisimple, $\exists K \lhd L$ compact

$\Rightarrow L = K \times H$ for some semisimple group H (Levi splitting)

$\hookrightarrow$

Furthermore, $\exists\, \tilde{\rho} : G \to H$,

section $\omega : G/\Gamma \to P$

such that

$$g \cdot \omega(b\Gamma) = \omega(gb\Gamma) \cdot \tilde{\rho}(g) \pmod{K}$$

WLOG: $\omega(e\Gamma) = e$

$$\Rightarrow \gamma \cdot \omega(e\Gamma) \overset{def}{=} \tilde{\omega}(\gamma) = \omega(e\Gamma) \cdot \rho(\gamma)$$

$$\left(\begin{tikzcd} G \arrow[r, "\tilde{\omega}"] \arrow[d] & P \\ G/\Gamma \arrow[ur, "\omega"'] & \end{tikzcd}\right).$$

$$\Rightarrow \gamma \cdot \omega(e\Gamma)$$

Zimmer

$$= \omega(\gamma\Gamma) \cdot \tilde{\rho}(\gamma)$$

$$= \omega(e\Gamma)\, \tilde{\rho}(\gamma) \pmod{K}$$

$$\Rightarrow \rho(\gamma) = \tilde{\rho}(\gamma) \pmod{K}$$

$$\Rightarrow \rho(\gamma) = \tilde{\rho}(\gamma)\, k(\gamma) \text{ for some } k : \Gamma \to K.$$

Comments on Zimmer Conjecture

Theorem (Brown, Fisher, Hurtado):

If $\Gamma \subseteq G$, $G = SL(d,\mathbb{R})$, $d \geq 3$ and $\Gamma \curvearrowright M$, $\dim(M) \leq d-2$, then the action is through a finite group.

Step 1: Show that Γ preserves a measure

Step 2: Look at the action on the frame bundle, must be (measurably) through a compact groups

Step 3: Look at actions preserving a measurable metric.