KV-Topics
1/23/25

(Today: Existence of Algebraic Hulls):

L-Reductions as Invariant Sections:

Lemma: If $\begin{array}{c} P \curvearrowleft H \\ \downarrow \\ X \end{array}$ is a principal H-bundle, $G \curvearrowright P$ by bundle automorphisms, and $Q \subseteq P$ is an L-reduction, then Q induces an H-equivariant map $\omega : P \to H/L$ $(\Rightarrow \Omega : X \to F_{H/L})$

$L \leq H$ and Q is a principal L-bundle.

Pf: Given $p \in P$, let $\hat{\omega}(p)$ denote any element of H such that $p\,\hat{\omega}(p) \in Q$. If ω_2 is any other such point in H, $p\,\hat{\omega}(p) = p\,\omega_2\,\ell$ for some ℓ
$\Rightarrow \hat{\omega}(p)^{-1}\omega_2 \in L$

↪

$\Rightarrow$ $\omega(p) \in H/L$ is well-defined and unique.

H-equivariance of ω:

$$p\,[\underbrace{h\,\omega(ph)}_{=\,\omega(p)}] \in Q \Rightarrow \boxed{\omega(ph) = h^{-1}\omega(p)}$$

Ex:

$$\begin{array}{c} \mathcal{F}(X) \curvearrowleft GL(d,\mathbb{R}) \\ \downarrow \\ X \end{array}$$

$SL(d,\mathbb{R})$-reduction gives rise to an invariant form,

$SO(d)$-reduction gives rise to an invariant metric

Towards the Algebraic Hull:

H = an $\mathbb{R}$-algebraic group

V = $\mathbb{R}$-variety (affine, projective, quasiprojective)

$H \curvearrowright V$ action defined on a Zariski-open set of V

(outside of a union of varieties.)

($\mathcal{Z}$-open $\Rightarrow$ open & dense in Euclid.)

Q: What are L-reductions of

$$\begin{array}{c} G \curvearrowleft H \\ \downarrow \\ G/H \end{array} \; ?$$

Ex: $H = GL(2,\mathbb{R})$, $V = \mathbb{R}^2$

$H \curvearrowright V$, $h \cdot v = hv$.

$\Rightarrow \exists$ exactly two orbits:

$\{0\}$, $\mathbb{R}^2 \setminus \{0\}$.

Ex: Borel subgroup $B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \curvearrowright V = \mathbb{R}^2$

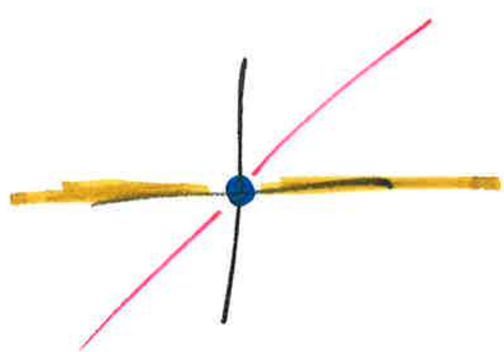
orbits: $\{0\}$,

$(\mathbb{R} \times \{0\}) \setminus \{0\}$,

$\mathbb{R}^2 \setminus (\mathbb{R} \times \{0\})$.

(~ Oseledets)

Ex: $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \subseteq GL(2,\mathbb{R})$



Theorem (Rosenlicht) : V decomposes as

H-invariant sets

really where H is defined action

$$V = M_0 \sqcup M_1 \sqcup \dots \sqcup M_\ell$$

such that $W_i = M_i \sqcup \dots \sqcup M_\ell$

is Zariski-closed in V

$M_i \subseteq W_i$ Zariski-open in W_i

M_i/H is a (smooth) variety over $\mathbb{R}$.

Ex : $B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$.

$$M_0 = 0$$

$$M_1 = (\mathbb{R} \times \{0\}) \setminus \{0\}$$

$$M_2 = \mathbb{R}^2 \setminus (\mathbb{R} \times \{0\}).$$

Cor: If $H \curvearrowright V$ is algebraic, then every orbit is a locally closed embedded submanifold. (Hx is open in $\overline{Hx}$.)

Def: Such actions will be called tame

Thm (Feres 1.2.2): An action is tame iff the orbit space $_H\backslash V$ is countably separated.

$\exists$ countable collection of open H-invariant subsets which separates points.

Reduction Lemma (Feres 6.4.1): Let $\begin{array}{c} P \curvearrowleft H \\ {\scriptstyle\pi}\downarrow \\ X \end{array}$ be a principal H-bundle, $H \curvearrowright V$ an algebraic action. Assume $\omega : P \to V$ is C^r $(r \geq 0)$, H-equivariant. Assume $G \curvearrowright P$ by bundle automorphisms,

and covers $G \curvearrowright X$ topologically transitive (also has a dense orbit).
If ω is G-invariant, then $\exists v_0 \in V$ such that $\omega^{-1}(v_0)$ is a C^r $\mathrm{Stab}(v_0)$-reduction of $G \curvearrowright P$.

Pf: Pick $x_0 \in X$ with a dense G-orbit. Let $p_0 \in P$ be any point covering x_0 and $v_0 = \omega(p_0)$.

$$Q = \omega^{-1}(v_0) = \{p \in P : \omega(p) = v_0\}$$

- Q is G-invariant: If $p \in Q$, $\omega(p) = v_0$ $\Rightarrow \omega(gp) = \omega(p) = v_0 \Rightarrow gp \in Q$.

$L = \mathrm{Stab}(v_0)$

$\hookrightarrow$

- Q is an L-sub-bundle:

$$\ell \in L,\ p \in Q \Rightarrow p\ell \in Q\ ?$$

$$\omega(p\ell) = \ell^{-1}\omega(p) = \ell^{-1}v_0 = v_0$$

$$h \in H,\ p \in Q,\ ph \in Q \Rightarrow h \in L\ ?$$

$$\omega(p) = v_0 = \omega(ph) = h^{-1}\omega(p)$$

$$v_0 = h^{-1}v_0 \Rightarrow h v_0 = v_0 \ \checkmark$$

- Q covers a dense set:

Q covers $G x_0$ ✓

- Q covers an open set:

By AG (Rosenlicht),

Hv_0 is open in $\overline{Hv_0}$

$$\mathrm{Im}(\omega) \subseteq \overline{Hv_0} \quad (\omega(G_{p_0}H) = Hv_0)$$

$\Rightarrow \omega^{-1}(Hv_0)$ is open. ✓

Prop: Suppose Q_1 and Q_2 are L_1- and L_2-reductions of $G \curvearrowright P \curvearrowleft H$, $P \to X$.
Then $\exists h_0 \in H$ and a $L_1 \cap (h_0 L_2 h_0^{-1})$-reduction.

(lemma)

Pf: By our first theorem, $\exists \omega_i : P \to H/L_i$ H-equivariant, G-invariant. Define $\omega = \omega_1 \times \omega_2$, $\omega(p) = (\omega_1(p), \omega_2(p))$.
Then ω is H-equivariant, G-invariant with respect to the diagonal H-action.

Note: $Q_1 = \omega^{-1}(V_1)$, $Q_2 = \omega^{-1}(V_2)$ for some V_1, V_2 such that

$$L_i = Stab(V_i)$$

$$\Rightarrow Stab(V_1, V_2) = L_1 \cap L_2.$$

$\hookrightarrow$

Wlog, $(v_1, v_2) \in \mathrm{Im}(\omega)$ by ~~translating~~ translating Q_2 by H.

Thm: $\exists$ minimal reduction, whose conjugacy class is unique. Denote it as the C^r-algebraic hull.

Ex: $f : \mathbb{T}^2 \to \mathbb{T}^2$. Anosov $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$-homotopy type, vol. preserving

suspend to get $g_t : X_f \circlearrowleft$.

$\Rightarrow$ C^0-algebraic hull $\cong \mathbb{R}$.

Pf: $\exists$ E^u, E^s-splitting (C^{0+}).

$Q \subseteq \mathcal{F}(\mathbb{T}^2)$

$Q = \{ \varphi : \mathbb{R}^2 \to T_x \mathbb{T}^2$ s.t. $\varphi(e_1) \in E^u, \varphi(e_2) \in E^s \}$

$\hookrightarrow$

C^r - alg. hull $\cong \mathbb{R}$

$\Updownarrow$

f C^∞ conj. to linear.

Ex: Suppose $G \curvearrowright H/\Gamma$ by translations,

Then C^∞-algebraic hull is $\cong G$.

$G \subseteq H$ both algebraic groups, G center-free.

Pf: $\varphi : \mathbb{R}^d \to T_e H$ arbitrary

$L = G \cong Ad(G)$

$$Q = \left\{ R_h \circ Ad(g) \circ \varphi \;\middle|\; \begin{matrix} g \in G \\ h \in H \end{matrix} \right\}.$$