KV-Topics

1/21/25

Today: Some algebraic geometry

<u>Associated Bundles</u>:

Let $\begin{array}{c} P \curvearrowleft H \\ \downarrow \\ X \end{array}$ be a principal $H$-bundle,

and $H \curvearrowright S$ be a left action.

<u>Common assumption</u>: $H \subseteq GL(d, \mathbb{R})$ is an $\mathbb{R}$-algebraic subgroup, $S = \mathbb{R}$-algebraic variety (projective)

Let $\overline{F} = P \times S \Rightarrow \exists\ \overline{F} \curvearrowleft H$ defined by

$$(p, s) \cdot h = (ph, h^{-1}s).$$

Claim: $\bar{F}/H$ is a fiber bundle over $X$ with fibers $S$.

$F = \bar{F}/H$ is a $C^\infty$ manifold because the $H$ action is free & proper.

...

(HW)

needed because ~~of~~ otherwise consider e.g.

$SL(2,\mathbb{R})/SL(2,\mathbb{Z})$ manifold.

$\Downarrow$

$SO(2)\backslash SL(2,\mathbb{R})/SL(2,\mathbb{Z})$ orbifold.

Ex: Consider $P = \mathcal{F}(X)$

= (full) frame bundle

$= \left\{ \varphi : \mathbb{R}^d \to T_xX \text{ for some } x \right\}$

= principal $GL(d,\mathbb{R})$-bundle

$S = \mathbb{R}^d$, $H \curvearrowright S$, $h \cdot s = hs$

$\Rightarrow$ Associated bundle $= TX$ (tangent bundle)

$\hookrightarrow$

$$\mathcal{F}(X) \times \mathbb{R}^d \to TX$$

$$(\varphi, v) \longmapsto \varphi(v) \in TX$$

$$(\varphi \circ A, A^{-1} v) \longmapsto \varphi(v)$$

is constant on $H$-orbits

$$\Rightarrow \quad F = \left(\mathcal{F}(X) \times \mathbb{R}^d\right)/_H \to TX$$

diffeo.

Rem: The dynamics does not act on the ~~f~~ orthonormal frame bundle (the action is not by isometries), hence the choice of the full frame bundle as opposed to ONF bundle.

Ex: $P = \mathcal{F}(X)$

$GL(d,\mathbb{R}) \curvearrowright Gr_k(\mathbb{R}^d) = \{k\text{-dim subspaces of } \mathbb{R}^d\}$

$\Rightarrow$ Associated bundle

$Gr_k(TX) = \{k\text{-dim subspaces of } T_xX \text{ for some } x.\}$

$(\varphi, E) \mapsto \varphi(E)$ descends to the factor by $H$.

Ex: $P = \mathcal{F}(X)$

determinantal line bundle $\det(TX) \to X$ ?

$GL(d,\mathbb{R}) \curvearrowright \mathbb{R}^*$

$A \cdot t = \det(A)\, t$

Associated bundle $= \{$top forms on $X\}$

$(\varphi, t) \mapsto t\, (\varphi^{-1})^*(dx_1 \wedge \dots \wedge dx_d)$

**Lemma**: There is a 1-1 ($C^\infty$ diffeo) correspondence between ($C^\infty$) sections $\Omega : X \to F$ and ($C^\infty$) $H$-equivariant maps $\omega : P \to S$ ($\leftrightarrow$ i.e. $\omega(ph) = h^{-1}\,\omega(p)$).

**Pf**: Let $\Omega : \underset{P/H}{X} \to \underset{(P\times S)/H}{F}$ be a section.

$$\boxed{\begin{tikzcd} P \arrow[d, "\pi"'] & \circlearrowleft H \\ X \end{tikzcd}}$$

Then $\Omega(x)$ is an $H$-orbit on $P\times S$. Fix $p \in P$. Then $\Omega(\pi(p))$ is an $H$-orbit in $(P\times S)/H$, i.e.,

$$\Omega(\pi(p)) = (q,s)H.$$

$\exists!$ $s' \in (q,s)H$ such that $(p,s') \in (q,s)H$. Define $\omega(p) = s'$

Let $\omega : P \to S$ be an $H$-equivariant map. Define $\bar{\Omega} : P \longrightarrow P \times S$

$$p \longmapsto (p, \omega(p))$$

By $H$-equivariance, $\bar{\Omega}$ descends to

$$\Omega : \underset{\substack{\| \\ X}}{P/H} \longrightarrow \underset{\substack{\| \\ F.}}{(P \times S)/H}$$

## Algebraic Hulls and Reductions:

(Main reason why we're using algebraic groups, as opposed to general Lie groups.)

**Def**: Let $H$ be a $\mathbb{R}$-algebraic group, and $P$ be a principal $H$-bundle. A (measurable/continuous/$C^r$) algebraic (principal) subbundle is ...

$\hookrightarrow$

a subset $Q \subseteq P$ covering
(a set of full measure / open dense subset) which is a principal $L$-bundle for some $\mathbb{R}$-algebraic subgroup $L \subseteq H$, where the action is just the restriction.

- If $G \curvearrowright P$ by automorphisms, a (meas / $C^r$) algebraic (principal) subbundle $Q \subseteq P$ is a (meas / $C^r$) $L$-reduction if it is $G$-inv.

Ex: $G \curvearrowright X$ preserves a smooth volume (on a dense open set)
$\Leftrightarrow \exists\, C^\infty$ $SL(d,\mathbb{R})$ reduction of $G \curvearrowright \mathcal{F}(X)$.

Pf of Ex: Assume $G$ preserves a smooth top form $\omega$. Let $Q \subseteq \mathcal{F}(X)$ denote the set of frames $(v_1, \_\_\_, v_d)$ such that

$$\omega(v_1, \_\_\_, v_d) = 1$$

$\Rightarrow$ $Q$ is a principal $SL(d, \mathbb{R})$-bundle. (HW: the other direction)

Ex: $G \curvearrowright X$ preserves a Riemannian metric (on an open dense set) iff $G \curvearrowright \mathcal{F}(X)$ admits an $O(d)$-reduction.

Def / Thm (Zimmer): If $G \curvearrowright P$ is an action by bundle automorphisms, the (meas / $C^1$) algebraic hull is an $L$-reduction which admits no proper reduction.

This is unique up to translations by $H$, i.e. if $Q_1, Q_2$ are algebraic hulls, then

$$Q_1 = Q_2 h \quad \text{for some } h \in H.$$

HW: Show that if a principal bundle $P$ has a global section $\sigma : X \to P$, $G \curvearrowright P$ and $\varphi$ is the induced cocycle (im$(\sigma)L$, where

$$\sigma(gx) = \sigma(x)\, \varphi(g,x),$$

then the algebraic hull is the smallest alg. subgroup $L \subseteq H$ such that $\varphi$ is cohomologous to a cocycle taking values in $L$.

HW*: Let $f: \mathbb{T}^2 \to \mathbb{T}^2$ be orientation-preserving Anosov diffeo. Show that the $C^\infty$ alg. hull of $\langle f \rangle$ is 1-dimensional $\Leftrightarrow$ $E^u, E^s$ are $C^\infty$

[$\Leftrightarrow$ $f$ is $C^\infty$ conj to linear]

HW**: Is there an Anosov diffeo $f: \mathbb{T}^2 \circlearrowleft$ whose algebraic hull is 2-dimensional?