KV-Topics

1/16/25

(Also follows from Kazhdan Property (T).)

Theorem (Infant Margulis): Let G be simple with $\operatorname{rank}_{\mathbb{R}}(G) \geq 2$, $\Gamma \leq G$ be a lattice, and $\rho : \Gamma \to \mathbb{R}$ be a homomorphism. Then $\rho \equiv 0$.

Bundle Construction: (~ suspension)

$$G \times \mathbb{R} \curvearrowleft \Gamma$$

$$(g, t) \cdot \gamma = (g\gamma, t + \rho(\gamma))$$

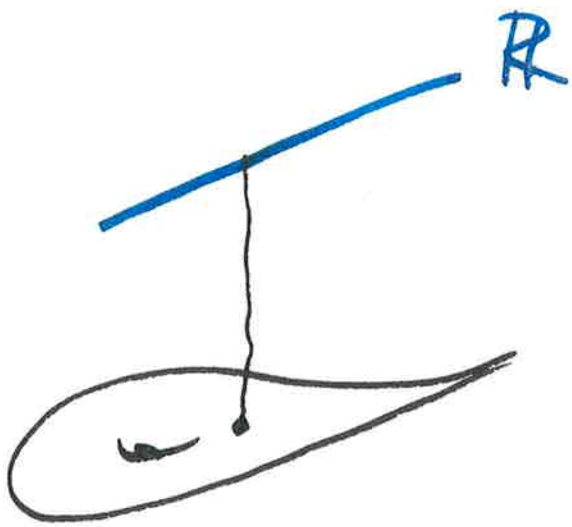
$X_\rho = (G \times \mathbb{R})/\Gamma$ is a principal $\mathbb{R}$-bundle over G/Γ.

Def: If H is a Lie group, a principal H-bundle over X is a fibration over X, $\mathcal{F}$, such that if $\pi : \mathcal{F} \to X$ is the factor map, $\pi^{-1}(\pi(y)) = yH$ for some right action $\mathcal{F} \curvearrowleft H$.

Lem (HW): Every principal $\mathbb{R}$-bundle has a (C^0) section.

It suffices to show that X_ρ has a G-invariant section.

$$\begin{array}{c} X_\rho \\ \downarrow \\ G/\Gamma \end{array}$$



Ex: Let X^d be a C^∞ manifold and

$$\mathcal{F}(X) = \{ (\varphi, x) \mid \varphi : \mathbb{R}^d \xrightarrow[\cong_{\mathbb{R}}]{} T_x X \}$$

is a principal $GL(d, \mathbb{R})$-bundle

$$(\varphi, x) \cdot A = (\varphi \circ A, x)$$

$\hookrightarrow$

Every C^∞ diffeo $f : X \to X$ lifts to a bundle automorphism

$$\tilde{f}(\varphi, x) = (df \circ \varphi, f(x))$$

<u>Note</u>: $\tilde{f}$ commutes with the H-action (i.e. it's a bundle auto)

<u>Setup</u>:

$$\begin{tikzcd} G \curvearrowright \mathcal{F} \curvearrowleft H \arrow[d] \\ X \end{tikzcd}$$

Assume $\exists$ section $\sigma : X \to \mathcal{F}$.

Any section must be of the form

$$\tau(x) = \sigma(x)\, h(x) \quad \text{for some } h : X \to H.$$

How does the G-action compare?

Write $\sigma(gx) = \sigma(x) \cdot \varphi(g, x)$

for some $\varphi(g, x) \in H$.

<u>Note</u>: σ is G-invariant

$\Leftrightarrow \varphi \equiv e_H$.

$$\tau(gx) = \sigma(gx) \cdot h(gx) = \sigma(x) \cdot \varphi(g, x) \cdot h(gx)$$

<u>Want</u> $\rightarrow \parallel$

$$\tau(x) = \sigma(x) \cdot h(x)$$

$\Leftrightarrow$

$$\boxed{\exists\, h \text{ such that } (h(x))^{-1} \cdot \varphi(g, x) \cdot h(gx) \equiv e_H}$$

Def: Let $G \curvearrowright X$. A matrix-valued <u>cocycle</u> over $G \curvearrowright X$ is a function

$$\varphi : G \times X \to H \subseteq GL(d, \mathbb{R})$$

such that

$$\varphi(g_1 g_2, x) = \varphi(g_2, x) \cdot \varphi(g_1, g_2 x).$$

- φ is called a <u>coboundary</u> if $\exists h : X \to H$ such that

$$(h(x))^{-1} \cdot \varphi(g, x) \cdot h(gx) \equiv e_H.$$

- φ, ψ are <u>cohomologous</u> if $\exists h : X \to H$ "transfer" such that

$$(h(x))^{-1} \cdot \varphi(g, x) \cdot h(gx) \equiv \psi(g, x).$$

$\Rightarrow$ It suffices to show that any $\mathbb{R}$-valued cocycle over $G \curvearrowright G/\Gamma$ is a coboundary. (⇐ reduction)

<u>Proving the Reduction</u>: (in 3 steps)

<u>Step 1</u>: Any $\mathbb{R}$-valued cocycle over $Z(A) \curvearrowright G/\Gamma$ is cohomologous to a constant, where A = $\mathbb{R}$-split Cartan subgroup. (Katok–Spatzier)

<u>Step 2</u>: If such a cocycle extends to a cocycle over the G-action, then it is a coboundary.

<u>Step 3</u>: Any A-invariant section is G-invariant.

Pf of Step 3 : (Steps 1 & 2 later)

Suppose $\sigma : G/\Gamma \to X_\rho$ is an A-invariant section. For $x \in G/\Gamma$, define $u_x(v)$ so that

$$\sigma(x) \cdot u_x(v) = \sigma(vx), \text{ for } v \in U_\lambda$$

(a root subgroup of G)

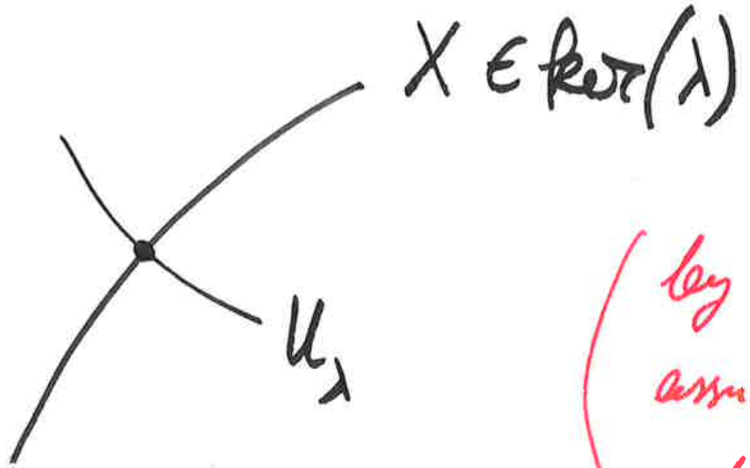
Recall: If $\mathfrak{g}$ a s.s. Lie algebra, and $\mathfrak{a}$ is an $\mathbb{R}$-split Cartan subalgebra then

$$\mathfrak{g} = Z(\mathfrak{a}) \oplus \bigoplus_{\lambda \in \Delta} E_\lambda,$$

where $\Delta \subseteq \mathfrak{a}^*$, $\lambda : \mathfrak{a} \to \mathbb{R}$ satisfies

$$\mathrm{ad}_X(U) = \lambda(X)U, \quad \forall U \in E_\lambda, X \in \mathfrak{a}$$

$U_\lambda = \exp\left(\bigoplus_{t>0} E_{t\lambda}\right)$ root subgroup.



(by the higher rank assumption, there is always a kernel.)

<u>Claim</u>: u_x is independent of x.

Using the claim,

$$u_x(v_1 v_2) = u_x(v_2)\, u_x(v_1)$$

$$= u_x(v_2) + u_x(v_1).$$

$\Rightarrow \forall$ root, the associated u is a homomorphism.

Lemma: Assume $\phi : \mathbb{R}^2 \to V$ is a map such that $\forall$ line $L \subseteq \mathbb{R}^2$: $\phi|_L$ is affine. Then ϕ is affine.

(horizontal or vertical)

($\sim$ Journé)

Ex: Consider $SL(3, \mathbb{R})$

$$\begin{pmatrix} * & \square & \boxed{} \\ & * & \\ & & * \end{pmatrix}$$

jointly integrable

(Later: $\mathbb{R}^2$ to be replaced by nilpotent, and affine with polynomial)

Pf of Claim (of Indep):

Consider $a \in \ker(\lambda)$. Then (really $a \in \exp(\ker(\lambda))$)

$$\sigma(ax)\, u_{ax}(v) = \sigma(vax) = \sigma(avx)$$

$$\Rightarrow \sigma(x)\, u_{ax}(v) = \sigma(vx)$$

$$\Rightarrow u_{ax}(v) = u_x(v).$$

$\hookrightarrow$

$x \mapsto u_x$ is constant along $\ker(\lambda)$.

$\ker(\lambda)$ has a dense orbit! (by Howe–Moore)

(Katok–Lewis)

Dynamical Normal Forms: Let $\mathcal{F}$ be a (orientable) 1D foliation of a manifold X, and $f : X \to X$ be a C^∞ diffeo which uniformly expands $\mathcal{F}$. Then $\exists\, \psi_x : \mathbb{R} \to X$ such that

$$\psi_{f(x)}^{-1} \circ f \circ \psi_x \text{ is linear.}$$

$$x \mapsto \psi_x \text{ is } C^0 \text{ into } C^\infty(\mathbb{R}; X).$$

Any $g : X \to X$ commuting with f satisfies

$$\psi_{g(x)}^{-1} \circ g \circ \psi_x \text{ is linear.}$$