KK Topics

1/14/25

Today:

Two Foundational Results & Raby Margulis.

$(\mathfrak{h} = \mathfrak{s})$ (over $\mathbb{C}$)

Thm (Lie): If $\mathfrak{h}$ is a solvable Lie algebra and $\pi : \mathfrak{h} \to \mathfrak{gl}(V)$ is a representation, then $\exists$ basis $v_1, \dots, v_d$ of V such that

$$\pi(\mathfrak{h}) \subseteq \begin{pmatrix} * & \cdots & * \\ 0 & \ddots & \vdots \\ 0 & 0 & * \end{pmatrix}.$$

Pf: Step 1: It suffices to show that $\pi(\mathfrak{h})$ has a common eigenvector.

$V_0 = V$

$v_0 =$ common eigenvector

$V_1 = V_0 / \langle v_0 \rangle$.

$\hookrightarrow$

$\pi(\mathfrak{h})$ descends to a rep on V_1

v_1 = common eigenvector of $\pi(\mathfrak{h}) \curvearrowright V_1$

$V_2 = V_1 / \langle v_1 \rangle$

etc.

$$\begin{array}{c} v_0 \\ v_1 \\ \vdots \\ v_{d-1} \end{array} \begin{pmatrix} * & * & \cdots & * \\ 0 & * & & \vdots \\ \vdots & 0 & \ddots & \\ 0 & 0 & & * \end{pmatrix}.$$

<u>Step 2</u>: Every solvable Lie algebra has an ideal of codim. 1.

$\left(\overset{\mathfrak{s}}{S} = \mathfrak{h} \right)$

$[\mathfrak{s},\mathfrak{s}] \neq \mathfrak{s}$, $\quad \mathfrak{s}/[\mathfrak{s},\mathfrak{s}]$ is abelian.

$p : \mathfrak{s} \to \underbrace{\mathfrak{s}/[\mathfrak{s},\mathfrak{s}]}$ onto

$\Rightarrow$ positive dim. $\to p^{-1}(W)$ is an ideal $\forall W$.

<u>Step 3</u>: If $\pi : \mathfrak{s} \to \mathfrak{gl}(V)$ is a rep, $\mathfrak{h} \leq \mathfrak{s}$ is an ideal of codim 1, and $\pi(\mathfrak{h})$ has a common eigenvector, then so does $\pi(\mathfrak{s})$.

<u>Pf</u>: v_0 = common evector for $\pi(\mathfrak{h})$

$\Rightarrow \exists\ \lambda : \mathfrak{h} \to \mathbb{C}$ homomorphism s.t.

$$\pi(H)\, v_0 = \lambda(H)\, v_0$$

Choose $X \in \mathfrak{s} \setminus \mathfrak{h}$.

$$v_0 \xrightarrow{\pi(X)} v_1 \xrightarrow{\pi(X)} v_2 \xrightarrow{\pi(X)} \cdots \xrightarrow{\pi(X)} \underset{\substack{\uparrow \\ \operatorname{span}\{v_0, \ldots, v_{\ell-1}\}.}}{v_\ell}$$

$H \in \mathfrak{h}$. $\quad \pi(H)\, v_{i+1} = \pi(H)\, \pi(X)\, v_i$

$$= \pi(X)\, \pi(H)\, v_i \mp \pi(\underbrace{[X, H]}_{\in \mathfrak{h} \text{ since } \mathfrak{h} \text{ is an ideal.}})\, v_i$$

$\hookrightarrow$

$$= \pi(X)\left[\lambda(H) v_i - w_{i-1}\right] \pm \left[\lambda([X,H]) v_i - \hat{w}_{i-1}\right]$$

(w_i, $\hat{w}_i$ to be determined)

$$= \left[\lambda(H) v_{i+1} - w_{i}\right] \pm \underbrace{\left(\lambda([X,H]) v_i - \hat{w}_{i-1}\right)}_{\in \operatorname{span}\{v_0, \dots, v_i\}}$$

$$\pi(H) v_{i+1} = \lambda(H) v_i + w_{i-1} \in \operatorname{span}\{v_0, \dots, v_{i-1}\}$$

Thus in this basis,

$$\pi(H) = \begin{pmatrix} \lambda(H) & & * \\ & \ddots & \\ 0 & & \lambda(H) \end{pmatrix}_{\ell}$$

$$\to \operatorname{tr}(\pi(H)) = \ell \cdot \lambda(H)$$

(Note: $\operatorname{tr}(\pi([X,Y])) = 0$.)

$\hookrightarrow$

$$tr\left(\pi(\underbrace{[X,H]}_{\in\mathfrak{h}})\right) = 0 = \ell\cdot\lambda([X,H])$$

$$\Rightarrow \lambda([X,H]) = 0.$$

$\Rightarrow \pi(H) = \lambda(H)\,\mathrm{id}$ on $\mathrm{span}\{v_0, \ldots, v_{\ell-1}\}$.

$\Rightarrow v_i$ is an evector of $\pi(X)$

Choose an eigenvector of $\pi(X)$ in

$\mathrm{span}\{v_0, \ldots, v_{\ell-1}\}$.

$\square$.

Bilinear Forms:

If $\beta : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is a symmetric bilinear form, let $b_{ij} = \beta(e_i, e_j)$.

$\Rightarrow \beta(v,w) = v^T B w$, where B is a symmetric matrix with $B_{ij} = b_{ij}$.

If $C : \mathbb{R}^d \circlearrowleft$ is a change of basis,

$$\beta(Cv, Cw) = \cancel{v^T C^T} \; v^T (C^T B C) w$$

<u>Thm</u>: If B is a symmetric matrix, then $\exists\, C \in SO(d)$ such that $C^{-1} B C = C^T B C$ is diagonal.

Thus up to a change of basis, every (nondegenerate) bilinear form is given as

$$\beta_{p,q}(v,w) = \sum_{i=1}^{p} v_i w_i - \sum_{i=p+1}^{p+q} v_i w_i$$

(p,q) is the <u>signature</u> of β.

Ex: $\beta(v,w) \leftrightarrow B = \begin{array}{cc} \begin{array}{ccc} p & p & q-p \end{array} & \\ \begin{pmatrix} 0 & I & 0 \\ I & 0 & 0 \\ 0 & 0 & I \end{pmatrix} & \begin{array}{c} p \\ p \\ q-p \end{array} \end{array}$.

HW: Check signature of β.

$so(p,q)$ = Lie alg. of matrices preserving

$so(p,q) = so(\beta)$

$SO(\beta) = \{A \mid A^T B A = B\}$.

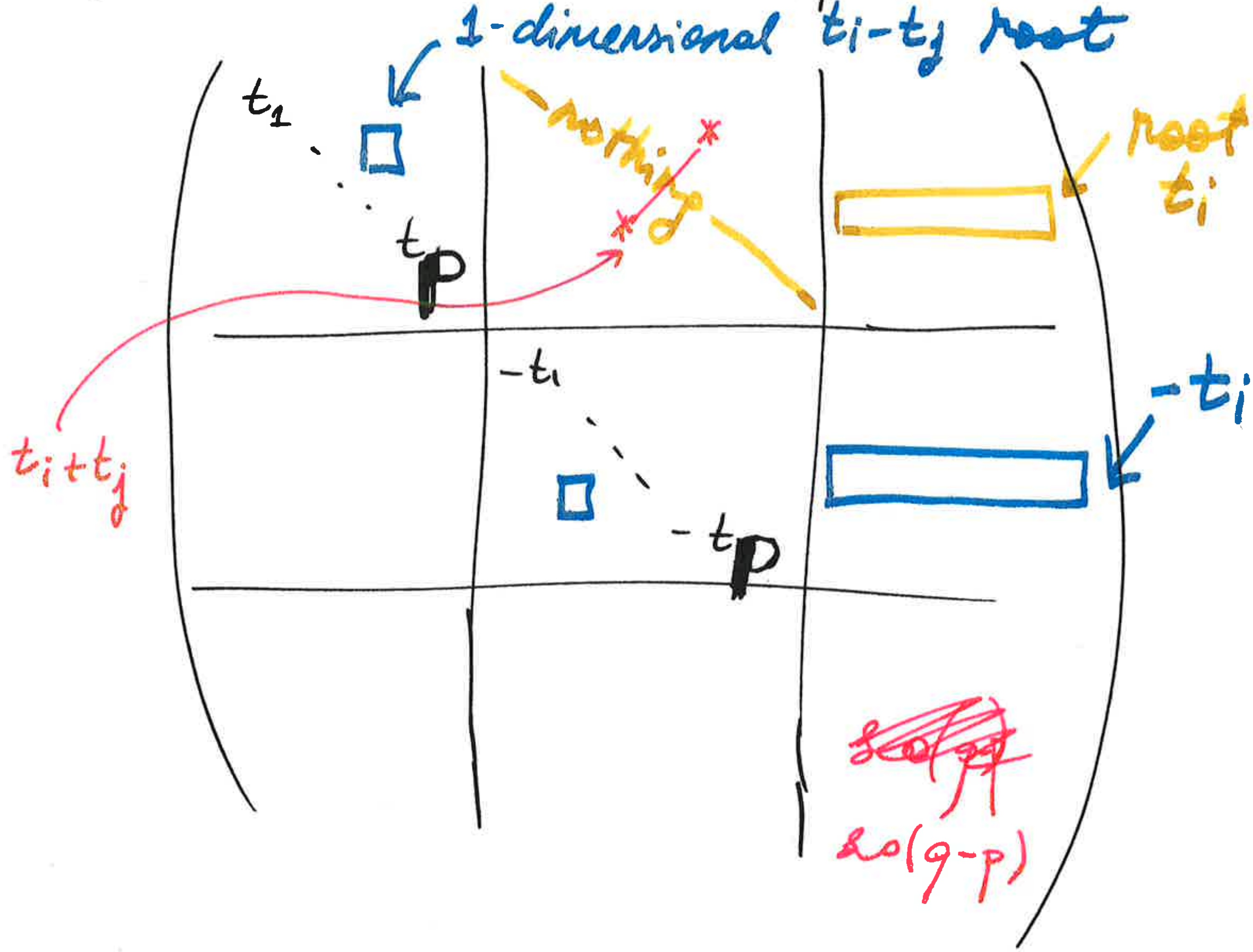
$A_t = \exp(tX)$.

(taking derivative) $\Rightarrow so(\beta) = \{X \mid X^T B + BX = 0\}$

$$= \left\{ \begin{array}{cc} \begin{array}{ccc} p & p & q-p \end{array} & \\ \left(\begin{array}{c|c|c} D & B & E \\ \hline C & -D^T & F \\ \hline -F^T & -E^T & Y \end{array}\right) & \begin{array}{c} p \\ p \\ q-p \end{array} \end{array} \;\middle|\; \begin{array}{l} D \in gl(p) \\ B, C \in so(p) \\ Y \in so(q-p) \end{array} \right\}$$

HW: Check this.

Nice subalgebras of $\mathfrak{so}(\beta)$:



t_i's $= \mathbb{R}$-split Cartan ~~subg~~ subalgebra.

t_i's $+$ maximal abelian subalgebra of $\mathfrak{so}(q)$ $=$ $\mathbb{C}$-split Cartan.

$\mathbb{R}$-split Cartan

$\cap \mathfrak{t}$ $\quad (q-p \geq 2)$

$\mathbb{C}$-split Cartan

$\cap \mathfrak{t}$ $\quad (q-p \geq 3)$

$Z(\mathbb{R}\text{-split Cartan})$

$\Rightarrow$ The set of roots:

$$\boxed{\Delta = \{t_i + t_j,\ t_i - t_j,\ t_i\}}.$$

Key Lemma: Let ~~$\phi : \mathbb{R}^k \to \mathrm{Aut}(\mathfrak{g})$~~ $\phi : \mathbb{R}^k \to \mathrm{Der}(\mathfrak{g})$ be a homomorphism s.t. $\phi(X)$ is diagonalizable over $\mathbb{R}$ $\forall X$.

Let $\lambda_1, \dots, \lambda_l : \mathbb{R}^k \to \mathbb{R}$ be the functionals giving the evals.

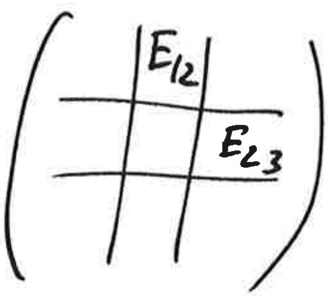
If E_{λ_i} is corresponding espace,

$$[E_{\lambda_i}, E_{\lambda_j}] \subseteq E_{\lambda_i+\lambda_j}$$

<u>Ex</u>: $SL(d, \mathbb{R})$.

roots $= \Delta = \{t_i - t_j\}$.

$$\Big[\underset{\substack{\uparrow \\ (t_i-t_j)}}{E_{ij}},\ \underset{\substack{\uparrow \\ (t_k-t_l)}}{E_{kl}}\Big] = \begin{cases} 0, & \text{if } j\neq k \text{ and } i \neq l. \\ E_{il}, & \text{if } j=k \text{ and } i\neq l \\ -E_{kj}, & \text{if } j \neq k \text{ and } i=l. \end{cases}$$



<u>Pf of Key Lemma</u>: Suppose $V \in E_{\lambda_i}$ and $W \in E_{\lambda_j}$. Then

$$\phi(X)([V,W]) \underset{\substack{\uparrow \\ (\text{derivation})}}{=} [\phi(X)V, W] + [V, \phi(X)W]$$
$$= [\lambda_i(X)V, W] + [V, \lambda_j(X)W] = (\lambda_i(X)+\lambda_j(X))[V,W]$$

Fix $X \in \mathfrak{a} \leftarrow \mathbb{R}$-split Cartan.

Let $E_X^u = \bigoplus_{\lambda(X)>0} E_\lambda \leftarrow$ root space.

(unstable space of X)

$\Rightarrow Y \in E_\lambda \subseteq E_X^u$

$\Rightarrow \mathrm{Ad}(\exp(tX))\,Y = e^{t\lambda(X)}\,Y$.

(exponential expansion)

Cor 1 : E_X^u is a subalgebra.

Cor 2 : E_X^u is nilpotent.

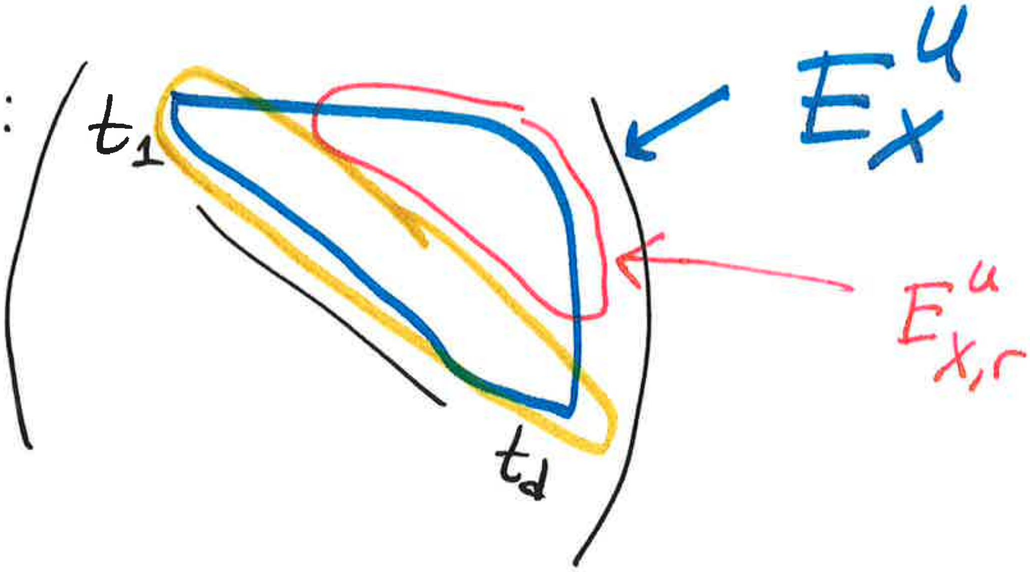
. Fix a rate $r>0$. Let $E_{X,r}^u = \bigoplus_{\lambda(X)\geq r} E_\lambda$

Then $E_{X,r}^u$ ~~also~~ is a nilpotent subalgebra.

fast distribution in E_X^u

* Fast distributions always integrate to foliations. Slow distributions typically fail to be integrable.

Ex:



HW: Check Key Lemma for

$$so(p,q) = so(\beta).$$

Thm (Infant Margulis): Let G be a semisimple Lie group with $\text{rank}_{\mathbb{R}}(G) \geq 2$. If Γ is a cocompact lattice in G and $\rho : \Gamma \to \mathbb{R}$ is a homomorphism, then $\rho \equiv 0$.

(dim of $\mathbb{R}$-split Cartan in $\mathfrak{g}$.)

Pf idea: Build a line bundle over G/Γ. (~ suspension).

References for Root Spaces:

- Helgason,
- Feres