KV-Topics
2/6/25

Setting:

$\omega : P \to V$

H-equivariant $H \curvearrowright V$ $\omega(ph) = h^{-1} \cdot \omega(p)$

A-invariant ($\mathbb{R}$-split Cartan in G)

$G \curvearrowright P \curvearrowleft H$
$\downarrow$
$G \curvearrowright (X, \mu)$

α = root of G

$\hookrightarrow U_\alpha$ = root subgroups

$\hookrightarrow G_\alpha = \langle U_\alpha, U_{-\alpha} \rangle$

$\Psi_\alpha : P \to \mathrm{Fun}(G_\alpha, V)$

$$\Psi_\alpha(p)(g) = \omega(g^{-1}p)$$

Ex: P = bundle of (homogeneous) frames on $T(G/\Gamma)$ $\quad G \curvearrowright G/\Gamma = X$

$$= \{ dL_{g_1} \circ dR_{g_2} \circ \varphi_0 \mid g_1, g_2 \in G \}$$

$\hookrightarrow$ for some fixed $\varphi_0 : \mathbb{R}^d \to T_e G$

$\mathfrak{h} \subseteq \mathrm{Lie}(G)$ a subalgebra

= distribution of right-invariant vector fields

$\Rightarrow \omega : P \longrightarrow \mathcal{G}r\left(\mathrm{Lie}(G), \dim(\mathfrak{h})\right)$

<u>Lemma</u>: $\Psi_\alpha(P)$ is a single H-orbit on $\mathrm{Fun}(G_\alpha, V)$ and

(for a set of full measure)

$H_\alpha(p) = \mathrm{Stab}_H\left(\Psi_\alpha(p)\right)$

is an algebraic group $\forall p \in P$.

Define $Q_{p,\alpha} = N_H\left(H_\alpha(p)\right) / H_\alpha(p)$

$\hat{W}_{p,\alpha} = H / H_\alpha(p) \cong \Psi_\alpha(P)$

$h \longmapsto hp$

$\circlearrowright$

Then

(1) $\{H_\alpha(p)\}_{p\in P}$ are all conjugate

2) $\exists\, W_{p,\alpha} \subseteq \tilde{W}_{p,\alpha}$ subvarieties

such that $\Psi_\alpha(p) \in W_{p,\alpha}$

and $W_{gp,\alpha} = W_{p,\alpha} \quad \forall g \in G_\alpha$

3) $\exists$ group transitive action

$Q_{p,\alpha} \curvearrowright W_{p,\alpha}$ (correction)

4) $\exists$ anti-homomorphisms

$\eta_{p,\alpha} : G_\alpha \longrightarrow Q_{p,\alpha}$ such that

$$\Psi_\alpha(gp) = \eta_{p,\alpha}(g) \cdot \Psi_\alpha(p).$$

Pf: Claim ★ : $\overset{\Psi_\alpha(P)}{\parallel}$ "single orbit"

$H \curvearrowright Fun(G_\alpha, V)$ is tame (i.e. orbit space is countably separated)

$(h\cdot\varphi)(g) = h\varphi(g)$

★ need regularity:

- C^r
- C^0
- measurable

Basic idea for claim:

Show that the orbits are locally closed.

↪

$\Psi_\alpha : P \to \mathrm{Fun}(G_\alpha, V)$

H-equivariance + tameness

$\Rightarrow \overline{\overline{\Psi}}_\alpha : P/H \to H \backslash \mathrm{Fun}(G_\alpha, V)$

is well-defined.

$\overline{\overline{\Psi}}_\alpha$ is A_α-invariant

$A_\alpha = \ker(\alpha)$
higher rank!

$\xRightarrow[\text{Howe-Moore}]{} \overline{\overline{\Psi}}_\alpha$ is constant a.e.

• Why is $H_\alpha(p)$ algebraic?

$$\mathrm{Stab}_H\left(\Psi_\alpha(p)\right) = \underbrace{\bigcap_{g \in G_\alpha} \underbrace{\mathrm{Stab}_H\left(\omega(gp)\right)}_{\text{algebraic}}}_{\text{algebraic}}$$

(Descending chain condition)

(1) Now if $\Psi_\alpha(q) = h\Psi(p)$,

$$\mathrm{Stab}(\Psi_\alpha(q)) \in h\,\mathrm{Stab}(\Psi_\alpha(p))\,h^{-1}$$

$\Rightarrow H_\alpha(p)$ is a.e. constant

(2) Define $W_{p,\alpha} = N_H(H_\alpha(p)) \cdot \Psi_\alpha(p)$

$$\tilde{W}_{p,\alpha} = H \cdot \Psi_\alpha(p).$$

$$\boxed{\underline{\text{Claim}}: W_{gp,\alpha} = W_{p,\alpha} \quad \forall g \in G_\alpha}$$

$$\Psi_\alpha(gp)(k) = \omega(k^{-1}gp) = \Psi_\alpha(p)(g^{-1}k)$$

$H_\alpha(p) = H_\alpha(gp)$ (same outputs)

↳ (3)

$$\boxed{\underline{\text{Need to show}}: \Psi_\alpha(gp) = h \cdot \Psi_\alpha(p) \text{ for some } h \in N_H(H_\alpha(p)).}$$ (4)

We know such an $h \in H$ exists.

Fix $k \in H_\alpha(p)$.

$$\boxed{\underline{\text{Want}}: h^{-1} k h \in H_\alpha(p)}$$

$$h^{-1} k h \, \Psi_\alpha(p)$$

$$h^{-1} k \, \Psi_\alpha(gp)$$

$$h^{-1} \Psi_\alpha(gp)$$

$$h^{-1} h \, \Psi_\alpha(p)$$

$$\Psi_\alpha(p) \quad \checkmark$$

G

For $g \in G_\alpha$ and $p \in P$, let

$\eta_{p,\alpha}(g) \in Q_{p,\alpha}$ be the unique element such that

$$\Psi_\alpha(gp) = \eta_{p,\alpha}(g) \cdot \Psi_\alpha(p)$$

Claim: $\eta_{p,\alpha}$ is an anti-homomorphism

aka.

$$\eta_{p,\alpha}(g_1 g_2) = \eta_{p,\alpha}(g_2) \cdot \eta_{p,\alpha}(g_1).$$

Pf of Claim:

Step 1: $\eta_{p,\alpha}$ has a cocycle property:

$$\eta_{p,\alpha}(g_1 g_2) \cdot \Psi_\alpha(p) = \Psi_\alpha(g_1 g_2 p)$$

$$= \eta_{g_2 p,\alpha}(g_1) \cdot \Psi_\alpha(g_2 p) \hookrightarrow$$

$= \dots$

$$\dots = \eta_{g_2 p, \alpha}(g_1) \cdot \eta_{p,\alpha}(g_2) \cdot \Psi_\alpha(p).$$

$$\Rightarrow \quad \eta_{p,\alpha}(g_1 g_2) = \eta_{g_2 p, \alpha}(g_1)\, \eta_{p,\alpha}(g_2)$$

$$= \eta_{p,\alpha}(g_2) \underbrace{\left(\eta_{p,\alpha}(g_2)^{-1} \eta_{g_2 p}(g_1)\, \eta_{p,\alpha}(g_2) \right)}$$

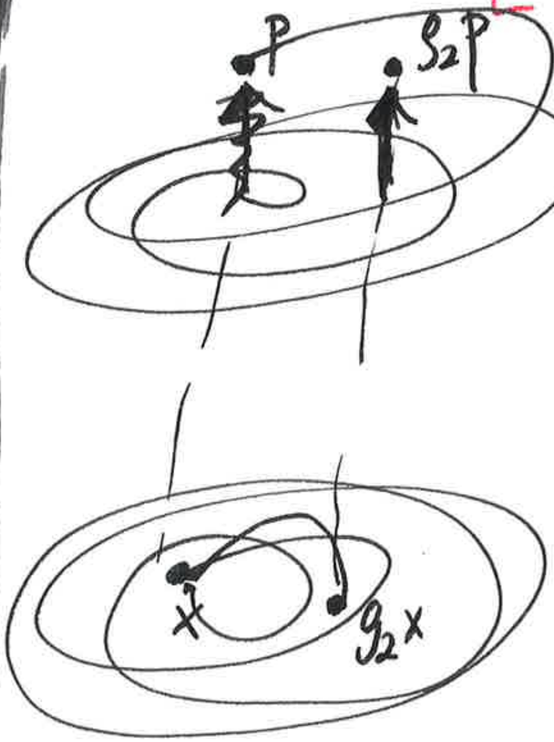
$$\Longrightarrow \eta_{p,\alpha}(g_1)\,?$$

<u>Step 2</u>: $\eta_{ap,\alpha}(g) = \eta_{p,\alpha}(g), \ \forall a \in A_\alpha$

$$\eta_{ap,\alpha}(g) \cdot \Psi_\alpha(ap) = \Psi_\alpha(gap)$$

$$= \Psi_\alpha(agp) = \Psi_\alpha(gp) = \eta_{p,\alpha}(g)\, \Psi_\alpha(p).$$

[∃ Mistake below.]



$a_k \in A_\alpha$ such that

$a_k x \to x$ (due to ergodicity Howe-Moore)

$a_k p h_k^{-1} \to g_2 p$

$$\eta_{a_k p h_k^{-1}, \alpha}(g) = \eta_{p h_k^{-1}, \alpha}(g)$$

$$= h_k \, \eta_{g_2 p, \alpha}(g) \cdot h_k^{-1}$$

<u>Note</u>: If $a_k p h_k^{-1} \to g_2 p$, then

$$\Psi_\alpha(a_k p h_k^{-1}) = \Psi_\alpha(p h_k^{-1})$$

$$= h_k \Psi_\alpha(p) \longrightarrow \Psi_\alpha(g_2 p)$$

$$= \eta_{p,\alpha}(g_2) \cdot \Psi_\alpha(p)$$

$\vdots$

$$h_k \to \eta_{p,\alpha}(g_2).$$

Ex. $SL(3,\mathbb{R}) \curvearrowright SL(3,\mathbb{R})/\Gamma$

$P=$ homogeneous frame bundle

$$\omega = \begin{pmatrix} * & \to & * \\ & \ddots & \downarrow \\ 0 & & * \end{pmatrix} - \text{section}$$

$$\mathrm{Stab}(\omega(g\Gamma)) = \mathrm{Ad}(g). \begin{pmatrix} * & \text{---} & * \\ & \ddots & | \\ 0 & & * \end{pmatrix}$$

$$H_\alpha(g\Gamma) = \mathrm{Stab}(\Phi_\alpha(g\Gamma)) = \bigcap_{k\in SL(2)} \mathrm{Ad}(k) \cancel{\mathrm{Ad}(g)} \begin{pmatrix} * & - & * \\ & \ddots & | \\ 0 & & * \end{pmatrix}$$

$$= \cancel{\mathrm{Ad}(g)} \left(\begin{array}{cc|c} t & 0 & * \\ 0 & t & * \\ \hline 0 & 0 & -2t \end{array}\right).$$

$$N_H(H_\alpha(g\Gamma)) = \left(\begin{array}{cc|c} * & * & * \\ * & * & * \\ \hline 0 & 0 & * \end{array}\right)$$

$$Q_{g\Gamma,\alpha} \cong SL(2).$$

$$a = \begin{pmatrix} t & & \\ & t & \\ & & -2t \end{pmatrix} \in A_\alpha \Rightarrow d_a = \mathrm{Ad}(a)$$

explodes, except on the isolated block.