KV-Topics
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Detour for Background

Theorem (Levi splitting): If $\mathfrak{g}$ and $\mathfrak{h}$ are Lie algebras, $\mathfrak{h}$ semisimple, and $\pi : \mathfrak{g} \to \mathfrak{h}$ surjective homomorphism. Then $\exists$ homomorphism $\sigma : \mathfrak{h} \to \mathfrak{g}$ such that $\pi \circ \sigma = id$.

$$\begin{tikzcd} \mathfrak{g} \arrow[d, "\pi"'] & \\ \mathfrak{h} \arrow[r, "id"'] & \mathfrak{h} \arrow[ul, "\exists \sigma"'] \end{tikzcd}$$

Facts:

- Every Lie algebra has a ~~sol~~radical i.e. the largest solvable ideal.

↳ **Exr:** If s_1 and s_2 are solvable ideals, then so is $s_1 \oplus s_2$.

• If $\mathfrak{s}$ is the solvradical, then $\mathfrak{g}/\mathfrak{s}$ is semisimple. (via the Killing form)

<u>Cor</u>: Every Lie algebra is a semidirect product $\mathfrak{l} \ltimes \mathfrak{s}$, where $\mathfrak{l}$ is semisimple and $\mathfrak{s}$ is solvable.

<u>Pf</u>: Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{s}$ be its solvradical. Then

$\pi : \mathfrak{g} \to \mathfrak{g}/\mathfrak{s}$ is surjective,

$\mathfrak{g}/\mathfrak{s}$ is semisimple,

so $\exists\ \sigma : \mathfrak{g}/\mathfrak{s} \to \mathfrak{g}$ homomorphism section

As a vector space, $\mathfrak{g} \cong \underbrace{\sigma(\mathfrak{g}/\mathfrak{s}) \oplus \mathfrak{s}}_{\text{subalgebras.}}$

$\hookrightarrow$

Define $\rho : \mathfrak{g}/\mathfrak{s} \to Der(\mathfrak{s})$

$$x \longmapsto ad_{\sigma(x)}$$

$$\Rightarrow \quad \mathfrak{g} \cong (\mathfrak{g}/\mathfrak{s}) \ltimes_\rho \mathfrak{s}.$$

<u>Important Note</u>: $\mathfrak{s}$ is unique, $\mathfrak{g}/\mathfrak{s}$ is unique, but $\sigma(\mathfrak{g}/\mathfrak{s})$ is not unique.

$\forall \sigma_1, \sigma_2 : \mathfrak{g}/\mathfrak{s} \to \mathfrak{g}$ hom. section, $\exists s \in S = \exp(\mathfrak{s})$ $\sigma_2 = Ad(s) \circ \sigma_1$.

Recall Zimmer Superrigidity Setting:

$G \curvearrowright P \curvearrowleft H$

$\downarrow$

$G \curvearrowright (X, \mu)$

- $\omega : P \to V$ is an H-equivariant map
- ω is A-invariant, where $A \subseteq G$ is an $\mathbb{R}$-split Cartan.

- $\Delta \subseteq A^*$ are the restricted roots of G (roots α such that 2α is not a root). (concern only for $SU(m,n)$)
- $\alpha \in \Delta \Rightarrow \exists\, \mathfrak{u}_\alpha \subset \mathfrak{g}$ (common eigenspaces of ad_x, $x \in Lie(A)$).
- $U_\alpha = \exp(\mathbb{R} \cdot \mathfrak{u}_\alpha (\oplus\, \mathfrak{u}_{2\alpha}))$ (if it ~~exists~~ exists).

e.g. in $SL(d, \mathbb{R})$,

$U_{\alpha_{ij}} = I + \mathbb{R} \cdot E_{ij}$.

- <u>Note</u>: If $A_\alpha = \ker(\alpha)$, then
  $$[A_\alpha, U_\alpha] = e.$$
- G_α = group generated by U_α and $U_{-\alpha}$.

  e.g. in $SL(d,\mathbb{R})$, $G_{\alpha_{ij}} \cong SL(2,\mathbb{R})$.

Define $\Psi_\alpha : P \to Fun(G_\alpha, V)$ by
$$\Psi_\alpha(p)(g) = \omega(g^{-1}p)$$

(restrict ω to G_α-orbits)

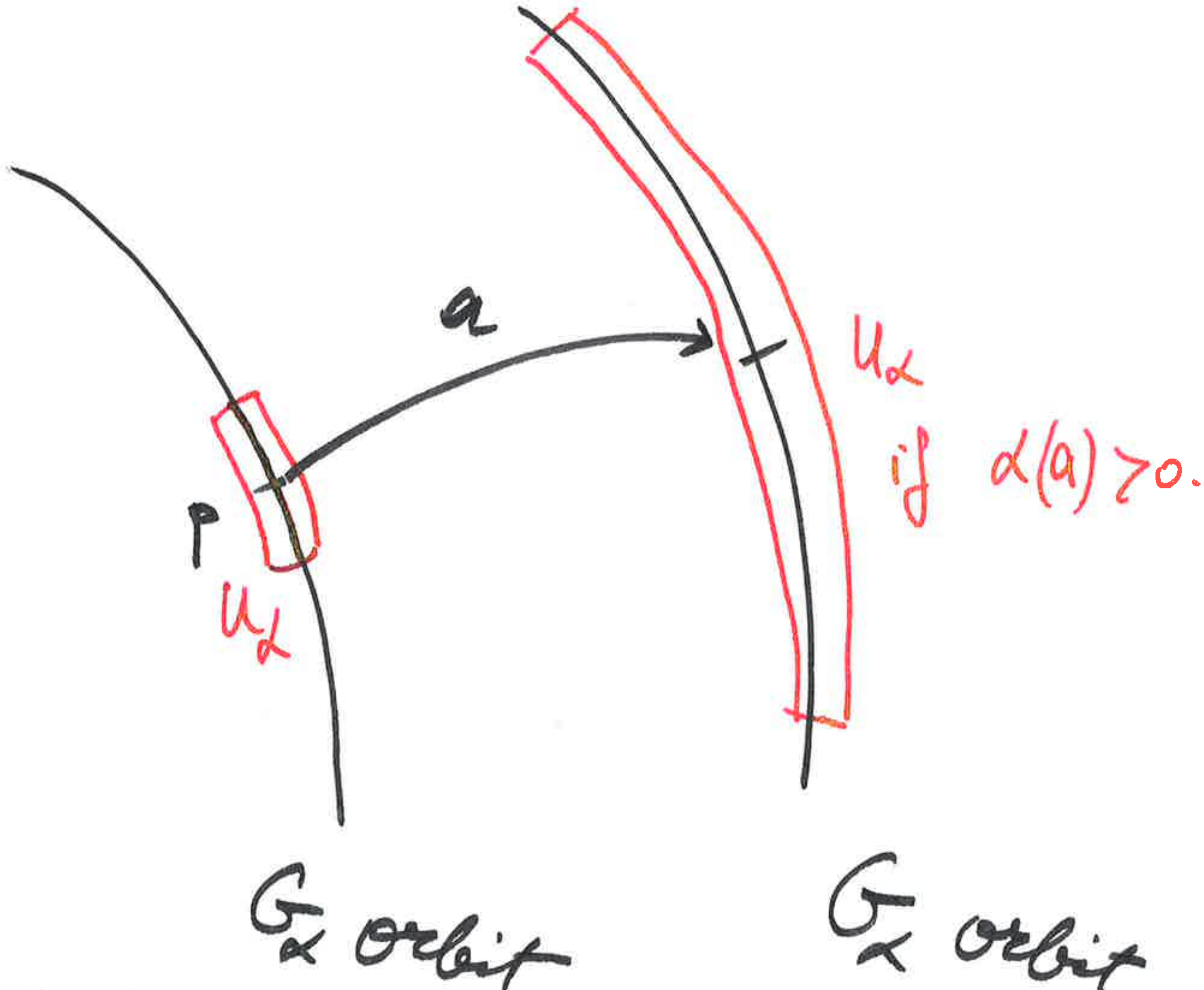
<u>Key Features</u>:

- $\Psi_\alpha(ph) = h^{-1}\Psi_\alpha(p) = L_{h^{-1}} \circ \Psi_\alpha(p).$
- (★) $\Psi_\alpha(ap) = \Psi_\alpha(p)$ $\forall a \in \ker(\alpha)$ but $a \notin A$ for every.

↪

- $\Psi_\alpha(gp) = \Psi_\alpha(p) \circ L_{g^{-1}} \quad \forall g \in G_\alpha$.

<u>More on ⊛</u> :



$$\Psi_\alpha(ap)(g) = \omega(g^{-1}ap) = \omega(ag^{-1}p)$$

$$\underset{\uparrow \atop A\text{-inv.}}{=} \omega(g^{-1}p) = \Phi_\alpha(p)(g).$$

Lemma: $\Psi_\alpha(P)$ is a single H-orbit on $\mathrm{Fun}(G_\alpha, V)$ and

$$H_\alpha(p) = \mathrm{Stab}_H(\Psi_\alpha(p)) \quad \forall p \in P$$

is an $\mathbb{R}$-algebraic subgroup.

Define for $p \in P$,

- $\tilde{W}_{p,\alpha} = H / H_\alpha(p) \cong \cancel{\Psi_\alpha(p)}$ $\cancel{\Psi_\alpha(P)}$ $\Psi_\alpha(p)$.
- $Q_{p,\alpha} = N_H(H_\alpha(p)) / H_\alpha(p)$.

Then 1) the groups $\{H_\alpha(p)\}_{p\in P}$ are conjugate.

2) $\exists$ subvarieties $W_{p,\alpha} \subseteq \tilde{W}_{p\alpha}$

$\hookrightarrow$

such that $\Psi_\alpha(p) \in W_{p,\alpha}$

$\forall \alpha\ p \in P$.

and $W_{gp,\alpha} = W_{p,\alpha}$, $\forall g \in G_\alpha$

3) $\exists$ (group) transitive action $Q_{p,\alpha} \curvearrowright W_{p,\alpha}$

4) $\exists$ homomorphisms

$$\eta_{p,\alpha} : G_\alpha \to Q_{p,\alpha} \text{ such that}$$

$$\Psi_\alpha(gp) = \cancel{\Psi_\alpha(p) \cdot \eta_{p,\alpha}(g)} \quad \forall g \in G_\alpha$$

$$= \eta_{p,\alpha}(g) \cdot \Psi_\alpha(p).$$

$\boxed{\text{Ultimately the } \cap \text{ of all } H_\alpha\text{'s will be compact.}}$

__Ex__: $SL(d,\mathbb{R}) \curvearrowright \underbrace{SL(d,\mathbb{R})/\Gamma}_{=X}$.

P = full frame bundle $= \mathcal{F}(X)$.

Fix $\underset{\text{regular}}{a} \in A$. Let ω be the section

$$x \in X \longmapsto E^s_a = T\left(\text{coset of} \begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix}\right)$$

$$\omega : P \to Gr_\ell(d^2-1).$$

$$\alpha = \alpha_{12} \Rightarrow G_\alpha = \left(\begin{array}{cc|ccc} * & * & & 0 & \\ * & * & & & \\ \hline & & 1 & & 0 \\ & 0 & & \ddots & \\ & & 0 & & 1 \end{array}\right)$$

$$\mathrm{Stab}_H(\omega(p)) \cong \begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ 0 & & * \end{pmatrix}.$$

Thus consider the stabilizer of the function Φ_α.!

↳

$$\mathrm{Stab}_H\left(\Phi_\alpha(p)\right) = \bigcap_{g\in G} \mathrm{Stab}_H\left(\omega(g^{-1}p)\right)$$

$$= \left(\begin{array}{c|c} \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} & * \\ \hline 0 & \begin{matrix} * & \cdots & * \\ & \ddots & \vdots \\ 0 & & * \end{matrix} \end{array}\right)$$

$$\cong H_\alpha(p).$$

$$N_H\left(H_\alpha(p)\right) = \left(\begin{array}{c|c} \overset{Q_{p,\alpha}}{\boxed{\begin{matrix} * & * \\ * & * \end{matrix}}} & * \\ \hline 0 & \begin{matrix} * & \cdots & * \\ & \ddots & \vdots \\ 0 & & * \end{matrix} \end{array}\right).$$

Remark: This example is almost universal. We'll be applying Oseledets to get an A-invariant section.

Next time: Pf of lemma.