KV-Topics
2/27/25

Rationality and Circular Orderings

Fix a semisimple Lie algebra $\mathfrak{g}$ with $\mathfrak{g} = \mathrm{Lie}(G)$, G $\mathbb{R}$-algebraic, $\mathfrak{a} \subseteq \mathfrak{g}$ is an $\mathbb{R}$-split Cartan subalgebra.

Def: $\mathfrak{b} \subseteq \mathfrak{a}$ is a <u>generic</u> <u>(2 dimensional)</u> <u>subalgebra</u> if

- $\dim(\mathfrak{b}) = 2$, and
- $\forall \alpha \in \Delta : \alpha|_{\mathfrak{b}} \neq 0$,
- $\forall \alpha, \beta \in \Delta : \alpha|_{\mathfrak{b}} = c\left(\beta|_{\mathfrak{b}}\right)$.

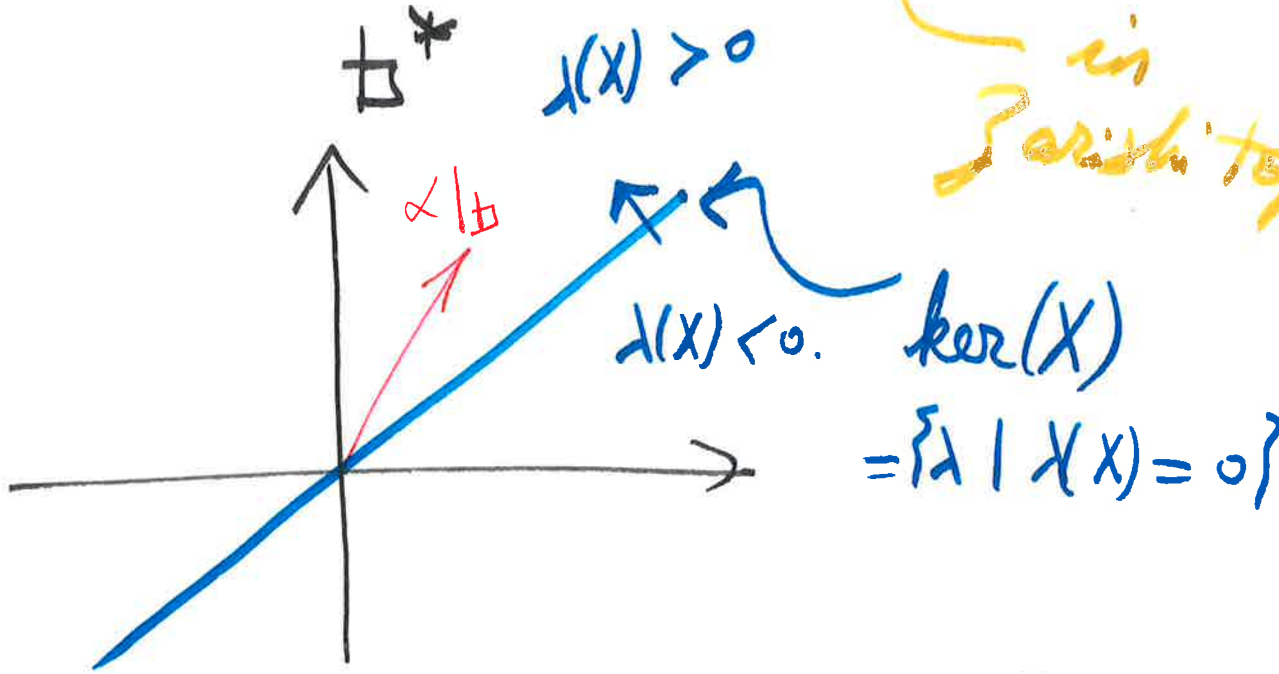
$$\Leftrightarrow \alpha = c\beta$$

HW: Generic subalgebras are <u>open and dense</u> in $\mathrm{Gr}(2, \mathfrak{a})$ in Zariski top.

- If $X \in \mathfrak{t} = (\mathfrak{t}^*)^*$, X is <u>generic</u> if
  $\alpha(X) \neq 0 \quad \forall \alpha \in \Delta$ and
  $\alpha(X) \neq \beta(X) \quad \forall$ distinct $\alpha, \beta \in \Delta$.

<u>HW</u>: The set of generic elements of a generic $\mathfrak{t}$ is <u>open & dense</u>.

is Zariski top.



Fix any $\lambda \in \ker(X) \subseteq \mathfrak{t}^*$.

Finally define $\measuredangle : \Delta^+(X) \to]0, \pi[$

$$\measuredangle(\beta) = \measuredangle(\beta, \lambda)$$

• The circular ordering on $\Delta^+(X)$ (relative to λ) is given by

$$\alpha_1 < \alpha_2 < \alpha_3 < \dots < \alpha_\ell$$

$$\measuredangle(\alpha_1) < \measuredangle(\alpha_2) < \dots < \measuredangle(\alpha_\ell).$$

Def: $U_i = \exp(\mathfrak{g}_{\alpha_i})$,

$$U_{[r]} = \exp\left(\bigoplus_{i=1}^{r} \mathfrak{g}_{\alpha_i}\right).$$

Theorem: (a) U_i and $U_{[r]}$ are $\mathbb{R}$-algebraic subgroups of $U_+(X)$.

(b). $U_\ell = U_+(X)$

(c). $U_{[r]}$ is normal in $U_{[r+1]}$

(d). $U_{[r]} \times U_{r+1} \to U_{[r+1]}$, $(u,v) \mapsto uv$ is ...

... a birational map ← (NOT a group homomorphism)

(e) • $\forall r$ $\exists X_r \in \mathfrak{t}$ such that

$$U_{[r]} = U_+(X) \cap U_+(X_r)$$

Ex: $\mathfrak{g} = \mathfrak{sl}(4, \mathbb{R})$

$$\mathfrak{t} = \left(\begin{array}{cc|cc} t+s & 0 & 0 & 0 \\ 0 & t-s & 0 & 0 \\ \hline 0 & 0 & -t+s & 0 \\ 0 & 0 & 0 & -t-s \end{array}\right)$$

$r_{12}|_{\mathfrak{t}} = 2s$ $\qquad$ $r_{23}|_{\mathfrak{t}} = 2t-2s$

$r_{13}|_{\mathfrak{t}} = 2t$ $\qquad$ $r_{24}|_{\mathfrak{t}} = 2t$ $\boxed{\text{nongeneric}}$.

$r_{14}|_{\mathfrak{t}} = 2t+2s$ $\qquad$ $r_{34}|_{\mathfrak{t}} = 2s$.

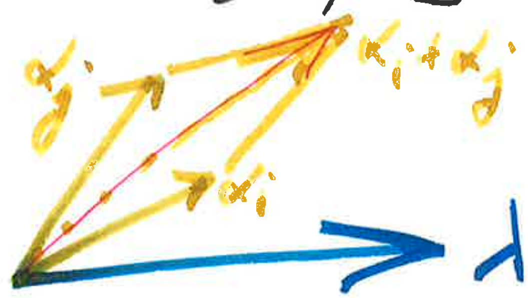
but a small perturbation of $\square$ is generic.

Pf of (a): Suppose that $X \in \mathfrak{g}_{\alpha_i}$, $Y \in \mathfrak{g}_{\alpha_j}$, $i, j \leq r$, then

(~ Lyapunov exponent)

$$[X, Y] \in \bigoplus_{k=1}^{r} \mathfrak{g}_{\alpha_k}$$

We know: $[X, Y] \in \mathfrak{g}_{\alpha_i + \alpha_j}$.



$$\measuredangle(\alpha_i + \alpha_j) \leq \max\{\measuredangle(\alpha_i), \measuredangle(\alpha_j)\}.$$

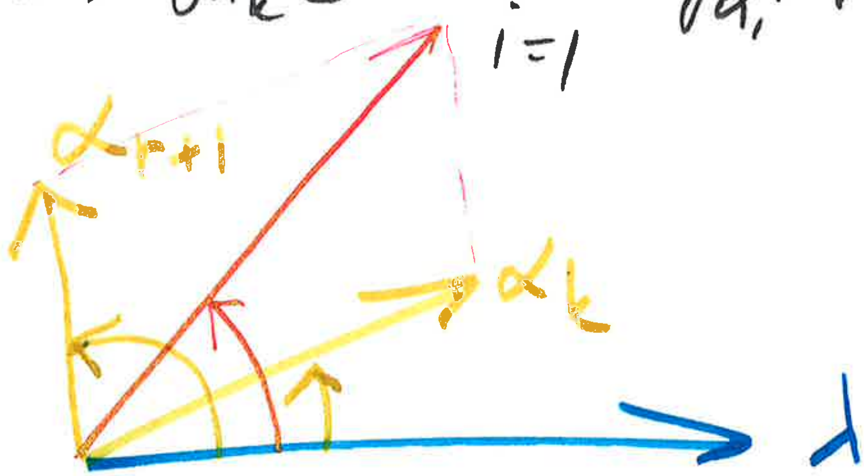
$$\left(\measuredangle(\alpha_j, \lambda) = \cos^{-1}\left(\frac{\langle \alpha_j, \lambda \rangle}{|\alpha_j| \cdot |\lambda|}\right)\right)$$

Pf of (b): Obvious.

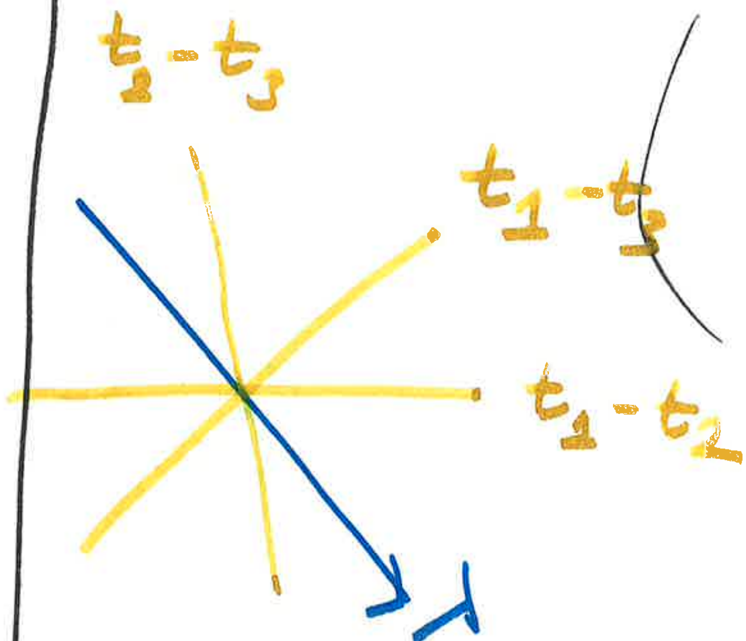
Pf of (c): We'll show $\mathcal{L}ie(\mathcal{U}_{[r]})$ is an ideal in $\mathcal{L}ie(\mathcal{U}_{[r+1]})$.

It suffices to show that

$$[\mathfrak{g}_{\alpha_{r+1}}, \mathfrak{g}_{\alpha_k}] \subseteq \bigoplus_{i=1}^{r} \mathfrak{g}_{\alpha_i} \quad (k=1,\dots,r).$$



Ex: $SL(3, \mathbb{R})$



$$\begin{pmatrix} & t_1 - t_2 & t_1 - t_3 \\ & & t_2 - t_3 \\ & & \end{pmatrix}$$

$$X = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{pmatrix}, \quad \lambda = 4t_1 - 5t_2 + t_2$$

$$u_{[1]} = \begin{pmatrix} 1 & * & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ -5 \\ 1 \end{pmatrix}$$

$$u_{[2]} = \begin{pmatrix} 1 & * & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$u_{[3]} = \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}$$

Pf of (d):

$$\begin{tikzcd}
(u,v) \arrow[r, mapsto] \arrow[d, mapsto, "\log"'] & uv \\
(\log(u), \log(v)) \arrow[ur, mapsto] &
\end{tikzcd}$$

$$\exp(\log(u)\ \log(v)) = \exp(p(\log(u), \log(v)))$$

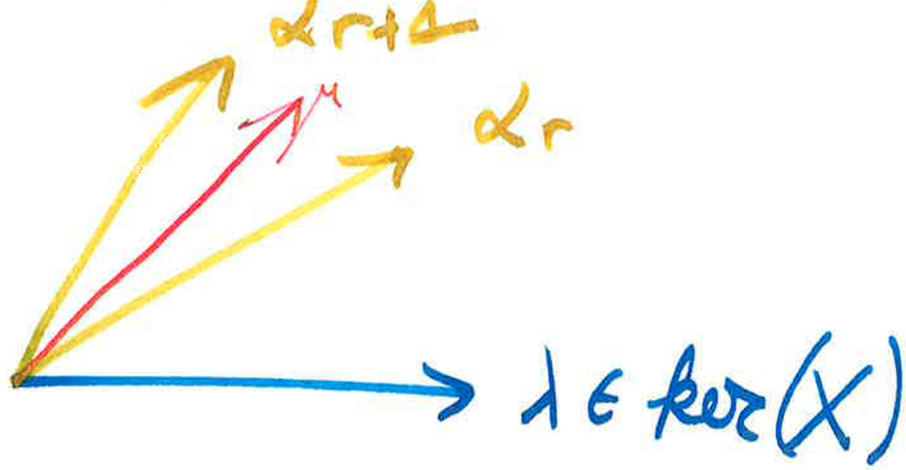
BCH formula.

BCH polynomial.

$$p(X,Y) = X + Y + \frac{1}{2}[X,Y] + \cdots$$

terminates (by nilpotence) and is invertible.

Pf of (e):



Pick μ between α_r and α_{r+1} in the circular ordering.

Let $X_r \in \mathfrak{t}$ be such that

$$\lambda(X_r) > 0 = \mu(X_r).$$

Exr: Show that

$$\{\alpha \in \Delta \mid \alpha(X), \alpha(X_r) > 0\} = \{\alpha_1, \dots, \alpha_r\}.$$

Margulis' Inductive Trick:

Assume $\varphi: U_+(X) \to V$ is (quasiprojective variety) measurable, and for Haar_{U_+}-a.e. $u \in U_+(X)$, $j = 1, \dots, \ell$

$\varphi|_{U_j \cdot u}$ coincides a.e. with a rational function.

Then φ coincides a.e. with a rational function

(Pf: next time.)