KV-Topics
2/25/25

Toward Zimmer super-rigidity,

Prepared ingredients

I) **The "long" lemma**: (done already)

$\omega: P \to V$ is A-inv., H-equiv..

If G is semisimple, α is a root, $G_\alpha = \langle U_\alpha, U_{-\alpha} \rangle$, then

" $\omega\big|_{G_\alpha}$ is induced from a homomorphism $G_\alpha \to Q_\alpha \ltimes W_\alpha \subseteq V$

II Homomorphisms from semisimple Lie groups are always $\mathbb{R}$-algebraic.

(Today)

III G_α is semisimple (& root classification for the layperson).

<u>Setting</u>: $\mathfrak{g}$ a Lie algebra over $\mathbb{C}$
(or split algebra over $\mathbb{R}$)

- $\mathfrak{a} \subseteq \mathfrak{g}$ is a <u>Cartan subalgebra</u> if
(or maximal torus)

1) $\mathfrak{a}$ is abelian
2) $\forall X \in \mathfrak{a}$: ad_X is diagonalizable
3) $\mathfrak{a}$ is maximal with 1) & 2).

__Construction__ of __Cartan subalgebras__:

$X \in \mathfrak{g}$ generic, (from an open dense subset)

$\mathfrak{a} = Z_{\mathfrak{g}}(X)$.

• The decomposition

$$\boxed{\mathfrak{g} = \mathfrak{a} \oplus \bigoplus_{\lambda \in \Delta} \mathfrak{g}_\lambda}$$

is called the __root space decomp__.

• $\Delta \subseteq \mathfrak{a}^*$ finite,

• $H \in \mathfrak{a}, X \in \mathfrak{g}_\lambda$

$\Rightarrow [H, X] = \lambda(H) \cdot X$.

Lem: $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subseteq \mathfrak{g}_{\lambda+\mu}$.

Lem: If $\lambda+\mu \neq 0$, then $\mathfrak{g}_\lambda \perp_K \mathfrak{g}_\mu$

(Killing form)

(i.e. $\forall X \in \mathfrak{g}_\lambda$, $\forall Y \in \mathfrak{g}_\mu$:
$K(X,Y) = 0$)

Before proof: $\boxed{K([A,B],C) = K(A,[B,C])}$ (associativity)

comes from

$$K([A,B],C) = \operatorname{tr}\left(\operatorname{ad}_{[A,B]}\operatorname{ad}_C\right)$$

$$= \operatorname{tr}\left([\operatorname{ad}_A, \operatorname{ad}_B]\operatorname{ad}_C\right)$$

use $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ to finish.

Pf of Lemma:

$$K(X,Y) = K\left(\frac{1}{\lambda(H)}[H,X],\, Y\right)$$

$$= -\frac{1}{\lambda(H)} K([X,H],\, Y)$$

$$= -\frac{1}{\lambda(H)} K(X,\, [H,Y])$$

$$= -\frac{1}{\lambda(H)} K(X,\, \mu(H)Y)$$

$$= -\frac{\mu(H)}{\lambda(H)} K(X,Y)$$

$$\Rightarrow -\frac{\mu(H)}{\lambda(H)} = 1$$

$\Rightarrow \lambda(H) + \mu(H) = 0$ unless $K(X,Y) = 0$.

Cor: $K|_{\mathfrak{a}}$ is nondegenerate.

Pf: View $\mathfrak{a} = \mathfrak{g}_0$, the 0-eigenspace.
Then by lemma, $\mathfrak{a} \perp_K \mathfrak{g}_\lambda$, $\forall \lambda \neq 0$.
Since K is non-degenerate,
$\forall X \in \mathfrak{a}, \exists Y \in \mathfrak{a} : K(X,Y) \neq 0$.

- By duality, $\forall \theta \in \mathfrak{a}^*$, $\exists H_\theta$ such that
$\theta(H) = K(H, H_\theta)$, $\forall H \in \mathfrak{a}$.

Theorem: (a) $\alpha \in \Delta \Rightarrow -\alpha \in \Delta$.
(b) If $X \in \mathfrak{g}_\alpha$, $Y \in \mathfrak{g}_{-\alpha}$, then
$[X, Y] = K(X,Y) \cdot H_\alpha$
(c) $K(H_\alpha, H_\alpha) \neq 0$
$\hookrightarrow$

(d) $\forall X \in \mathfrak{g}_\alpha$, $\exists Y \in \mathfrak{g}_{-\alpha}$ such that if $\hat{H}_\alpha = \dfrac{2H_\alpha}{K(H_\alpha, H_\alpha)}$, then

$$\left.\begin{aligned} &[X, Y] = \hat{H}_\alpha, \\ &[\hat{H}_\alpha, X] = 2X, \\ &[\hat{H}_\alpha, Y] = -2Y. \end{aligned}\right\} \quad (\mathfrak{sl}_2\text{-triple})$$

Pf: (a) Since $\mathfrak{g}_\alpha \perp_K \mathfrak{a}$ and $\mathfrak{g}_\alpha \perp_K \mathfrak{g}_\beta$ $\forall \beta \neq -\alpha$, if K is nondegenerate, $-\alpha \in \Delta$.

(b) $\forall H \in \mathfrak{a}$, $K(H, [X,Y]) \overset{\text{(associativity)}}{=} K([H,X], Y)$

$= \alpha(H)\, K(X,Y) = K(H_\alpha, H) \cdot K(X,Y)$

$\hookrightarrow$

$$= K\left(K(X,Y)H_\alpha, H\right)$$

$$= K\left(H, K(X,Y)H_\alpha\right)$$

$$\Rightarrow [X,Y] = K(X,Y)H_\alpha$$

by nondegeneracy of $K|_{\mathfrak{g}}$.

(c) Assume, for a contradiction, that

$$K(H_\alpha, H_\alpha) = 0$$

$$\Rightarrow \alpha(H_\alpha) = 0$$

$$\Rightarrow [H_\alpha, X] = 0, \quad \forall X \in \mathfrak{g}_\alpha$$

$$[H_\alpha, Y] = 0, \forall Y \in \mathfrak{g}_{-\alpha}$$

(H_α commutes with $\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}$)

$\exists X \in \mathfrak{g}_\alpha, \exists Y \in \mathfrak{g}_{-\alpha}$ such that

$$K(X,Y) = 1 \underset{(b)}{\Rightarrow} [X,Y] = H_\alpha.$$

$\hookrightarrow$

$\Rightarrow \langle X, Y, H_\alpha \rangle \underset{Lie}{\cong} (\mathfrak{Heis}) = \begin{pmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix}$

Since $H_\alpha = [X, Y]$ and belongs to an ad-nilpotent subalgebra, it is nilpotent

But H_α is also semisimple;

nilpotent + semisimple $\Rightarrow H_\alpha = 0$, a contradiction.

Exr: If $\mathfrak{h} \subseteq \mathfrak{g}$ is nilpotent, $X = [Y, Z]$, $Y, Z \in \mathfrak{h}$ $\Rightarrow \mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$ is nilpotent.

(d) Fix $X \in \mathfrak{g}_\alpha$, and choose $Y \in \mathfrak{g}_{-\alpha}$ such that

$$K(X,Y) = \frac{2}{K(H_\alpha, H_\alpha)} = \frac{2}{\alpha(H_\alpha)}.$$

$$\Rightarrow [X, Y] = K(X, Y) H_\alpha = \frac{2}{\alpha(H_\alpha)} H_\alpha = \hat{H}_\alpha$$

$$[\hat{H}_\alpha, X] = \frac{2}{\alpha(H_\alpha)} [H_\alpha, X] = \frac{2}{\alpha(H_\alpha)} \operatorname{ad}(H_\alpha) X$$
$$= 2X.$$

Similarly, $[\hat{H}_\alpha, Y] = -2X.$

- Fix a generic $H \in \mathfrak{a}$

  (i.e., $\alpha(H) \neq 0, \forall \alpha \in \Delta$,
  $\alpha(H) \neq \beta(H), \forall \alpha \neq \beta \in \Delta$)

Let $\Delta^+(H) = \{\alpha \in \Delta \mid \alpha(H) > 0\}$
(positive roots) (~ unstable dirs)

$$\Delta_S(H) = \left\{ \alpha \in \Delta^+(H) \,\middle|\, \begin{array}{l} \alpha \neq \beta_1 + \beta_2, \\ \beta_1, \beta_2 \in \Delta^+(H) \end{array} \right\}$$
(simple roots)

Thm: $\Delta_S(H)$ "generates" $\Delta^+(H)$

Ex: $\mathfrak{sl}(3,\mathbb{R})$, $H = \begin{pmatrix} x_1 & & \\ & x_2 & \\ & & x_3 \end{pmatrix}$

$x_1 > x_2 > x_3$

$\Rightarrow \Delta^+(H) = \{t_1 - t_2,\ t_2 - t_3,\ t_1 - t_3\}$

$\Delta_S(H) = \{t_1 - t_2,\ t_2 - t_3\}$

Thm: $\Delta_S(H)$ is a basis of $\mathfrak{a}^*$.

List the elements of $\Delta_S(H)$:

$\{\alpha_1, \dots, \alpha_\ell\}$. Define the matrix

$$C_{ij} = \frac{2K(H_{\alpha_i}, H_{\alpha_j})}{K(H_{\alpha_i}, H_{\alpha_i})}.$$

(Cartan matrix).

Thm: The Cartan matrix $C = (C_{ij})$ has integer entries, and determines $\mathfrak{g}$ up to isomorphism.

Ex: $\mathfrak{sl}(3, \mathbb{R}) = \mathfrak{g}$.

$$\Delta_S = \{t_1 - t_2,\ t_2 - t_3\}$$

$$\cancel{[H_{t_1-t_2}, E_{12}] = 2(t_1 - t_2)}$$

$$K\left(H_{t_1-t_2}, \begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{pmatrix}\right)$$

$$= t_1 - t_2$$

> **Exr**: If G is a matrix group, then $K(X, Y) = \operatorname{tr}(XY)$.

$$\Rightarrow \quad H_{t_1 - t_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

so

$$H_{t_2-t_3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

$$C_{12} = \frac{2\,K\left(H_{t_1-t_2}, H_{t_1-t_3}\right)}{K\left(H_{t_1-t_2}, H_{t_1-t_2}\right)} = \frac{2\cdot(-1)}{2} = -1.$$

$$C_{11} = C_{12} = 2 \Rightarrow C = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

Ex: $\mathfrak{sl}(d,\mathbb{R}) \doteq \mathfrak{g}$.

$$\Rightarrow C = \begin{pmatrix} 2 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2 \end{pmatrix}.$$

Ex: $\mathfrak{g} = \mathfrak{so}(2,2)$.

$$\left(\begin{array}{cc|cc} t_1 & \square & 0 & \triangle \\ -\square & t_2 & \triangle & 0 \\ \hline 0 & -\triangle & -t_1 & -\square \\ -\triangle & 0 & \square & -t_2 \end{array}\right)$$

say H has $t_1 > t_2 > 0$.

$$\Delta^+(H) = \{t_1 - t_2,\ t_1 + t_2\} = \Delta_S(H).$$

$$H_{t_1-t_2} = \left(\begin{array}{cc|cc} 1/2 & 0 & 0 & 0 \\ 0 & -1/2 & 0 & 0 \\ \hline 0 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & 1/2 \end{array}\right).$$

$$H_{t_1+t_2} = \left(\begin{array}{cc|cc} 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ \hline 0 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & -1/2 \end{array}\right)$$

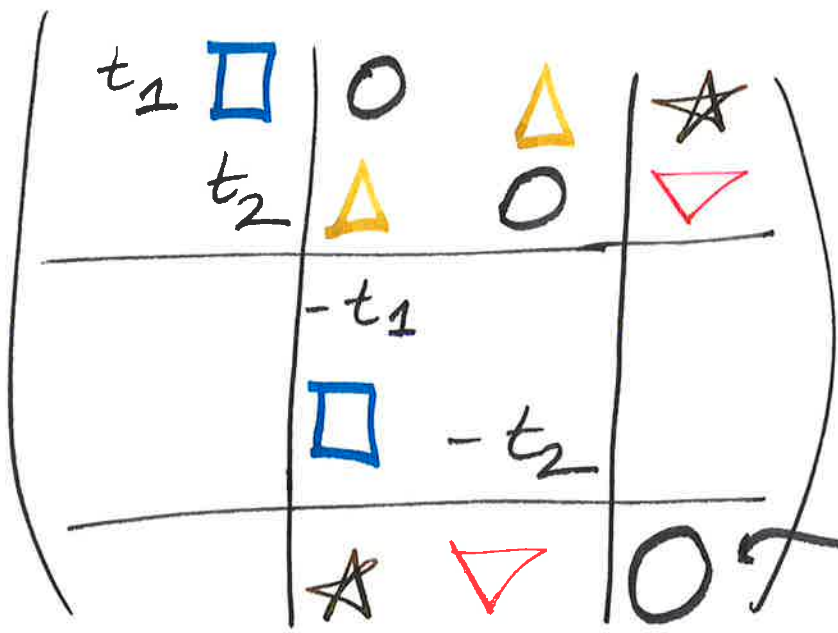
$\hookrightarrow$

$$\Rightarrow \kappa(H_{t_1-t_2}, H_{t_1+t_2}) = 0.$$

$$\Rightarrow \quad C = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

$$\Rightarrow \quad \mathfrak{so}(2,2) = \mathfrak{sl}(2) \oplus \mathfrak{sl}(2).$$

Ex : $\mathfrak{g} = \mathfrak{so}(2,3)$



$\mathfrak{so}(3-2) = \mathfrak{so}(1) = 0$

H be so that $t_1 > t_2 > 0$.

$$\Delta^+(H) = \{t_1, t_2, t_1 - t_2, t_1 + t_2\}$$

$$\Delta_S(H) = \{t_2, t_1 - t_2\}$$

$$H_{t_2} = \left(\begin{array}{cc|cc|c} 0 & & & & \\ & 1/2 & & & \\ \hline & & 0 & & \\ & & & -1/2 & \\ \hline & & & & 0 \end{array}\right)$$

$$H_{t_1 - t_2} = \left(\begin{array}{cc|cc|c} 1/2 & & & & \\ & -1/2 & & & \\ \hline & & -1/2 & & \\ & & & 1/2 & \\ \hline & & & & 0 \end{array}\right).$$

$$c_{12} = \frac{2 \ (-1/2)}{1/2} = -2.$$

$$c_{21} = \frac{2 \cdot (-1/2)}{1} = -1$$

$$\Rightarrow \quad C = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$$