KV-Topics
2/20/25

Digression: Partial Hyperbolicity

(toward Wilkinson visit)

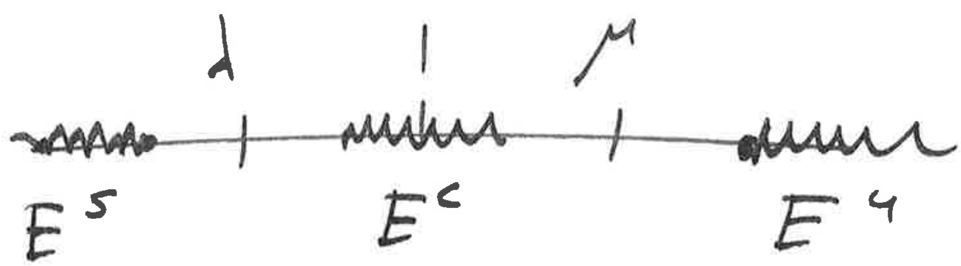
Def: $f: X \to X$ is partially hyperbolic if

$TX = E^u \oplus E^c \oplus E^s$ and

$\exists\, \lambda < 1 < \mu$ such that

$$\| df|_{E^s} \| < \lambda < \| df^{-1}|_{E^c} \|^{-1} <$$

$$\| df|_{E^c} \| < \mu < \| df^{-1}|_{E^u} \|^{-1}$$



Ex: $\varphi_t : X \to X$ Anosov flow

$f = \varphi_1$ is partially hyperbolic w/r/t
$TX = E^s_\varphi \oplus \langle v \rangle \oplus E^u_\varphi$.

Ex: Assume $A \in SL(3, \mathbb{Z})$ has eigenvalues $\lambda_1 < 1 < \lambda_2 < \lambda_3$ with eigenspaces E_1, E_2, E_3, respectively. Then

$A : \mathbb{T}^3 \circlearrowleft$ is ~~partial~~ partially hyperbolic w/r/t

$$E^s = E_1$$
$$E^c = E_2$$
$$E^u = E_3$$

Ex: Let $f: X \to X$ be Anosov / PH, P be a principal K-bundle over X with K compact, and $F: P \to P$ be a bundle automorphism covering f.

Prop: Any such F is PH w/r/t

$$\hat{E}^u \oplus TK \oplus \hat{E}^s \quad (\text{for } f \text{ Anosov})$$

$$\hat{E}^u \oplus (TK \oplus \hat{E}^c_f) \oplus \hat{E}^s \quad (\text{for } f \text{ PH}).$$

sub-ex 1: $P = X \times K$.

$$F(x, k) = (f(x), k\,\varphi(x))$$

for some $\varphi: X \to K$. C^∞

Sub-ex 2 : Consider H = Heisenberg

$$= \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

$$\mathrm{Lie}(H) = \left\{ \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \right\}$$

$$= \left\{ \left(\begin{pmatrix} x \\ y \end{pmatrix}, z \right) \right\}.$$

$$\left[\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, z_1 \right), \left(\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, z_2 \right) \right] = \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \det \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} \right]$$

HW: Check this

$\Rightarrow \forall A \in SL(2,\mathbb{R})$, $\left(\begin{pmatrix} x \\ y \end{pmatrix}, z \right) \mapsto \left(A \begin{pmatrix} x \\ y \end{pmatrix}, z \right)$

is an auto of Heis.

$\hookrightarrow$

Lift A to H?

$$\begin{array}{ccc} H & \xrightarrow{\hat{A}} & \cancel{H}\ H \\ \Big\downarrow \log & & \Big\uparrow \exp \\ \mathrm{Lie}(H) & \xrightarrow{A} & \mathrm{Lie}(H) \end{array}$$

$$\exp\left(\begin{pmatrix} x \\ y \end{pmatrix}, z\right) = \begin{pmatrix} 1 & x & z + \frac{xy}{2} \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

$$\cancel{\log\left(\begin{pmatrix} x \\ y_1 \end{pmatrix}, z\right)}$$

$$\log\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = \left(\begin{pmatrix} a \\ b \end{pmatrix}, c - \frac{ab}{2}\right).$$

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \overset{\hat{A}}{\longmapsto} \begin{pmatrix} 1 & & \\ 0 & 1 & \\ 0 & 0 & 1 \end{pmatrix}$$

$c - \frac{ab}{2} + \frac{1}{2}\left(\left(A\begin{pmatrix} a \\ b \end{pmatrix}\right)_1 \cdot A\begin{pmatrix} a \\ b \end{pmatrix}_2\right)$

$A\begin{pmatrix} a \\ b \end{pmatrix}$

e.g. $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

$$\hat{A} \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2a+b & c+a^2+ab+\frac{b^2}{2} \\ 0 & 1 & a+b \\ 0 & 0 & 1 \end{pmatrix}$$

- $\hat{A}$ preserves $\Gamma = \left\{ \begin{pmatrix} 1 & k & m/2 \\ 0 & 1 & \ell \\ 0 & 0 & 1 \end{pmatrix} \middle| \begin{array}{l} k, \ell, m \\ \in \mathbb{Z} \end{array} \right\}$

  whenever $A \in SL(2, \mathbb{Z})$

lattice in H.

(Heisenberg nilmanifold)

$\Rightarrow \hat{A} : H/\Gamma \circlearrowleft$

<u>Note</u>: H/Γ is an $\mathbb{R}/\frac{1}{2}\mathbb{Z}$ - principal bundle over $\mathbb{T}^2$ and $\hat{A}$ is a bundle automorphism.

$\hookrightarrow$

If (v_1, v_2) and (w_1, w_2) are s/u vectors for A, respectively, then

$$E^s_{\hat{A}} = \left\{ \begin{pmatrix} 0 & v_1 & 0 \\ 0 & 0 & v_2 \\ 0 & 0 & 0 \end{pmatrix} \right\}$$

$$E^u_{\hat{A}} = \left\{ \begin{pmatrix} 0 & w_1 & 0 \\ 0 & 0 & w_2 \\ 0 & 0 & 0 \end{pmatrix} \right\}$$

$$E^c_{\hat{A}} = \left\{ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}.$$

HW: Let ω be a 2-form on $\mathbb{R}^{2n}$ and $A \in Sp(\omega)$ (i.e., $\omega(Av, Aw) = \omega(v, w)$) $\cap SL(2n, \mathbb{R})$. Define H to be the (generalized) Heisenberg group, with Lie algebra $\mathbb{R}^{2n} \times \mathbb{R}$

and bracket

$$[(v_1, t_1), (v_2, t_2)] = (0, \omega(v_1, v_2)).$$

Show • $H \cong \left\{ \begin{pmatrix} 1 & x_1 \cdots x_n & s \\ 0 & I & \begin{matrix} y_1 \\ \vdots \\ y_n \end{matrix} \\ 0 & 0 & 1 \end{pmatrix} \right\}$

- $\exists$ lattice $\Gamma \subseteq H$ preserved by $\hat{A}$,
- $\hat{A} \curvearrowright H/\Gamma$ is a principal bundle auto, covering $A \curvearrowright \mathbb{T}^{2n}$.

Sub-ex 3: G semisimple, $a \in A$ is <u>not</u> regular, $K \subseteq Z_G(a)$.

Then

$$L_a : G/\Gamma \circlearrowleft \text{ covers}$$

$$L_a : K \backslash G/\Gamma \circlearrowleft .$$

e.g.

$$\left(\begin{array}{c|c|c} 2 & & \\ \hline & 1/2 & \\ \hline & & \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} \end{array}\right) \in SO(n,1)$$

orthonormal

frame flow

$\downarrow$

geodesic flow

$\hookrightarrow$

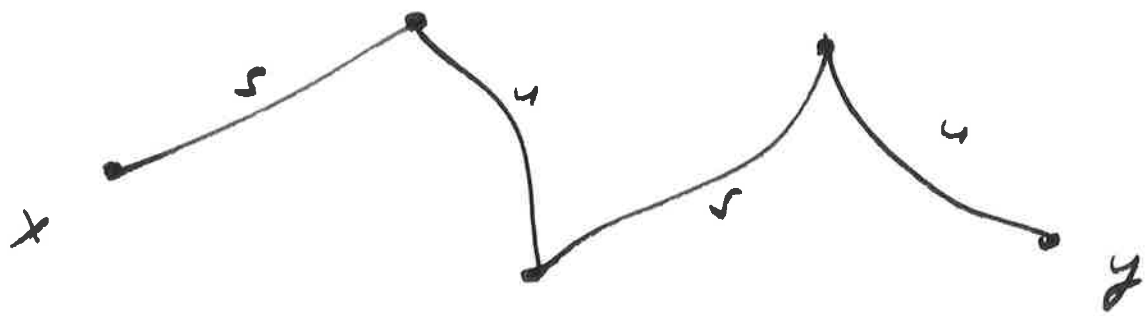
e.g.

$$\begin{pmatrix} 2 & & \\ & \tfrac{1}{2} & \\ & & \underbrace{\begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix}}_{SO(d-2)} \end{pmatrix} \in SL(d,\mathbb{R}).$$

Open Q: Which nilmanifolds support Anosov diffeos?

nilpotent group modulo a lattice.

Def: An ***su-path*** is a path γ such that γ is continuous and is piecewise contained in an s-leaf or u-leaf. The set of endpoints of su-paths starting at x is called the ***accessibility class*** of x



<u>Fact</u>: Accessibility classes partition X.

- f is called <u>accessible</u> if $\exists$ only one accessibility class

<u>Conj</u>: Accessibility classes ~~at~~ are immersed submanifolds for any partially hyperbolic diffeo.

<u>Theorem</u> (Burns–Wilkinson) If $f: X \circlearrowleft$ is volume preserving and accessible, then it is ergodic (in fact, Bernoulli).

(Pf is based on the Hopf argument).

Theorem (Wilkinson; Livshitz): Let $F : P \circlearrowleft$ be a principal bundle automorphism as before, covering a PH / Anosov base

(accessible and transitive).

Wilkinson Livshitz.

Then $\exists$ an F-invariant measurable section $\Leftrightarrow$

$\exists$ F-invariant

PH: C^0-section ((C^r along s/u)

Anosov: C^r-section

(Invariance Principle.)

Lemma/Exe: $F : P \circlearrowleft$ is C^0 conjugate

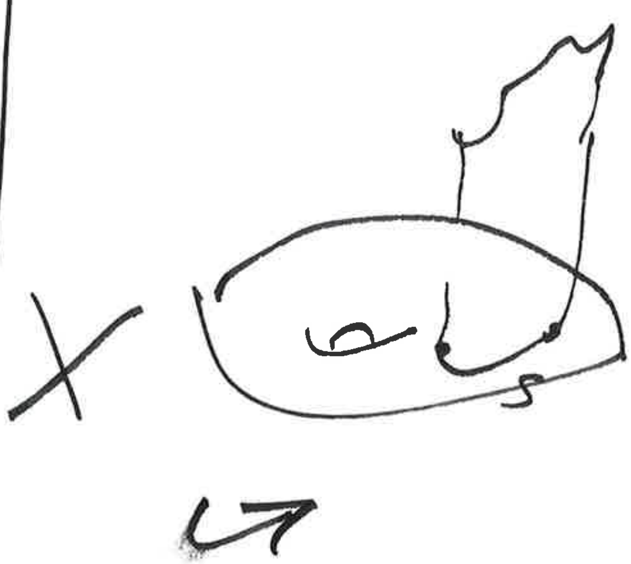
As a direct product

$$\tilde{F}(x,k) = (f(x), k)$$

$\Leftrightarrow$ $\exists$ F-invariant section of $P \to X$.

Def/Lemma: Let F, f be as before.

Then $W^s_F(p)$ covers $W^s_f(\pi(p))$.



Let $y \in W^s_f(x)$.

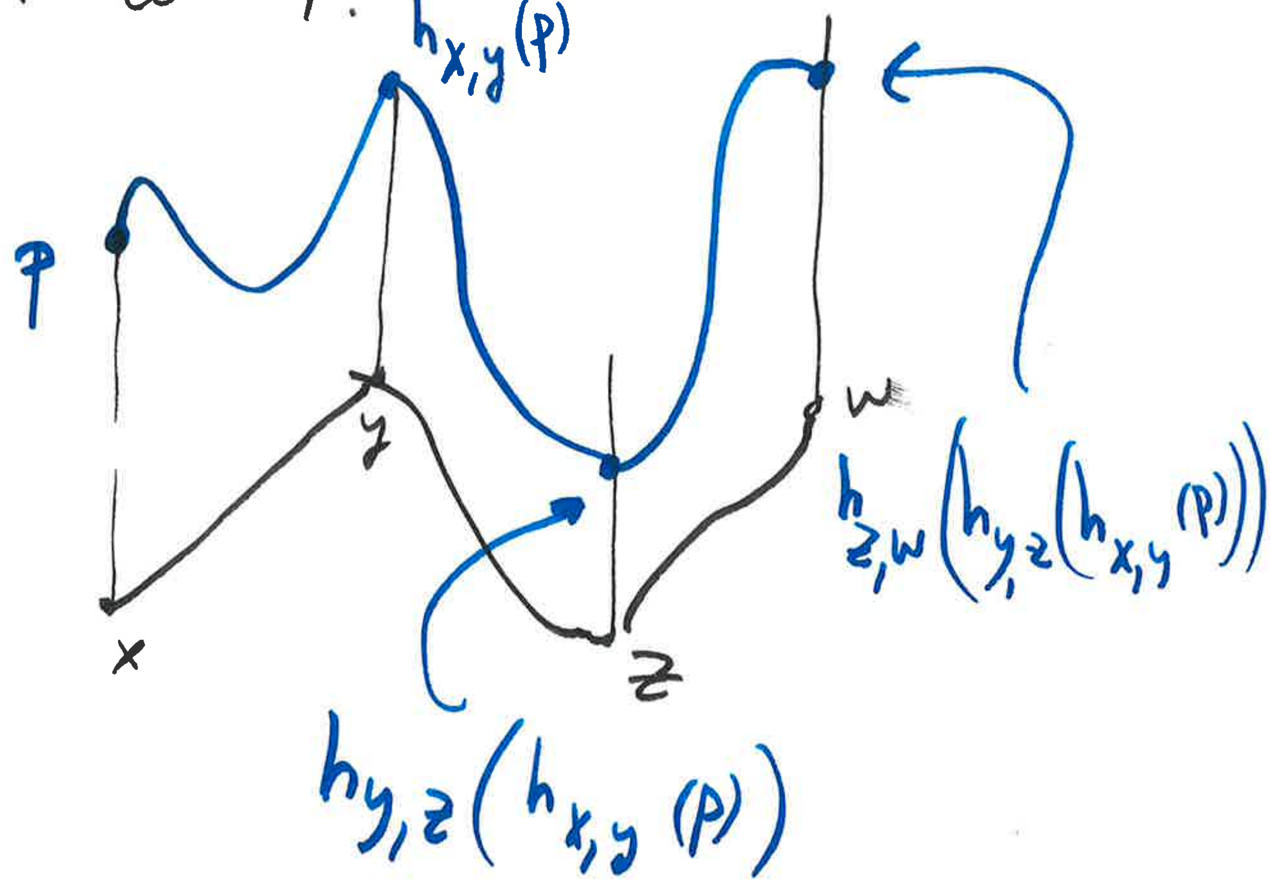
Then define

$$h^s_{x,y} : \pi^{-1}(x) \to \pi^{-1}(y)$$

$$p \mapsto W^s_F(p) \cap \pi^{-1}(y)$$

(stable holonomy).

This allows us to lift s/u paths on X to P.



If γ_x is an su-path at x, $\pi(p) = x$, let γ_p denote its lift to p.

An su-path is called a <u>cycle</u> if its starting and end pt is the same.

Theorem (Katok - Kononenko):

Let P, F, f be as above, assume f is accessible. Then $\exists F$-invariant section $\Leftrightarrow \forall$ su-cycle γ_x, γ_p is an su-cycle.