KV-Topics
2/18/25

Today: Borel & Parabolic subgroups.

<u>Question/Open Problem</u>: Topological Howe-Moore.

<u>Def</u>: $f: X \to X$ continuous is called <u>nonwandering</u> if $\forall x \in X$ and $x \in U \underset{\text{open}}{\subseteq} X$, $f^{n_k}(U) \cap U \neq \emptyset$ for some sequence $n_k \nearrow \infty$.

(i.e. $NW(f) = X$)

<u>Ex</u>: $t \mapsto t+1$ on $\mathbb{R} \Rightarrow NW = \emptyset$ (also every point is wandering).

Ex: North-South pole map

$\Rightarrow NW = \{N, S\}$.

$Q^{***}$: Suppose $\alpha: G \curvearrowright X$ is a $C^0$ $G$-action, $X$ compact and $\forall g \in G$, $NW(\alpha(g)) = X$. Then $\alpha$ is topologically mixing in the following sense:

$\forall U, V \subseteq X$ open, $\exists$ compact $K \subseteq G$ such that $\forall g \notin K$, $\alpha(g)U \cap V \neq \emptyset$.

Rem 1: If $\alpha$ preserves a fully supported inv. measure, then this follows from the usual Howe-Moore.

Rem 2 : The nonwandering ~~set~~ condition is necessary to rule out boundary actions (e.g. $SL(d,\mathbb{R}) \curvearrowright S^{d-1}$)

## Parabolic subgroups & Boundary actions:

(Recall)

Def : A semisimple Lie group $G$ is called $\underline{\mathbb{R}\text{-split}}$ if $Z_G(A) = A$ (up to finite groups)

Ex: • $SL(d,\mathbb{R})$ is $\mathbb{R}$-split

• $SL(d,\mathbb{C})$ is not $\mathbb{R}$-split

• $SO(m,n)$ is $\mathbb{R}$-split $\Leftrightarrow |m-n| \leq 1$.

We'll focus on $\mathbb{R}$-split groups first

Thm: Every complex group has a unique $\mathbb{R}$-split ~~Real~~ real form.

(up to $\mathbb{R}$-isomorphism)

Ex:

$$\begin{tikzcd} & SL(n,\mathbb{C}) \arrow[dl, "\mathbb{R}\text{-split}"'] \arrow[dr, "\text{compact}"] & \\ SL(d,\mathbb{R}) & & SU(n) \end{tikzcd} \qquad \begin{tikzcd} & SO(2n,\mathbb{C}) \arrow[dl, "\mathbb{R}\text{-split}"'] \arrow[dr, "\text{compact}"] & \\ SO(n,n) & & SO(2n,\mathbb{R}). \end{tikzcd}$$

For split groups, we have the root decomposition:

$$\boxed{\mathrm{Lie}(G) = \mathrm{Lie}(A) \oplus \bigoplus_{\lambda \in \Delta} \mathfrak{g}_\lambda}$$

where $\Delta \subseteq \mathrm{Lie}(A)^*$

(i.e. there is no $Z_G(A)$ term)

<u>Def</u>: Let $G$ be $\mathbb{R}$-split. A <u>Borel subgroup</u> of $G$ relative to some $a \in A$ such that $\lambda(a) \neq 0$ $\forall \lambda \in \Delta$ is defined as

$$B = \exp(\mathfrak{b}), \text{ where}$$

$\hookrightarrow$

Zhengi Wang papers for $SO(n,n)$

$$\mathfrak{B} = \mathscr{L}ie(A) \oplus \bigoplus_{\lambda(a) > 0} \mathfrak{g}_\lambda$$

$\mathfrak{B}$ = "weak-unstable distribution of $a$"

= "center-unstable distribution of $a$"

= "things that are neutral ~~of~~ or expanded by $a$"

Ex: $B = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \subseteq SL(3, \mathbb{R})$

Borel subgroup

Thm: $B$ (Borel subgroup) is unique up to conjugacy by $\{g \in G \mid gAg^{-1} = A\} / Z_G(A) = W$

Weyl group

$= A$ since $A$ is $\mathbb{R}$-split.

Ex: For $SL(d)$ $W = S_d$.

**Def**: $Q \subseteq G$ is called <u>parabolic</u> if it is algebraic and contains a Borel subgroup.

**Theorem**: If $H \subseteq G$ is algebraic and $G$ is semisimple and algebraic, then

- $G/H$ is quasi-projective
- $G/H$ is projective $\Longleftrightarrow$ $H$ is parabolic.

**Important Consequence of Theorem:**

$G/H$ is compact $\Leftrightarrow$ $H$ is parabolic.
(for $H$ algebraic)

## Connection with the Iwasawa Decomposition:

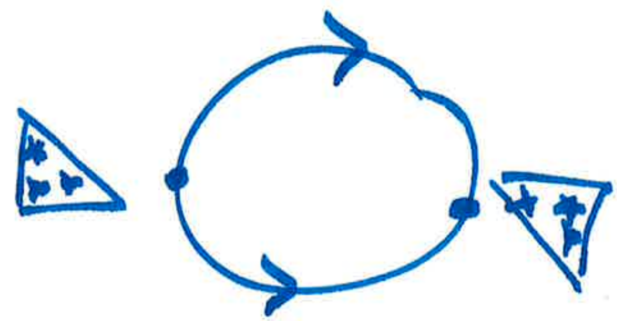
$G$ semisimple

$\Rightarrow \quad G = \underbrace{K}_{\text{Compact}}\underbrace{AN}_{\text{Borel}}$ (topologically)

<u>Ex</u>: $SL(2,\mathbb{R})$, $B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$.

$SL(2,\mathbb{R}) \curvearrowright SL(2,\mathbb{R})/B \cong \mathbb{RP}^1$

is the projective action. (HW).



G semisimple,
Q parabolic.

Theorem: $G \curvearrowright G/Q$ does not preserve an invariant probability measure (unless $G = Q$)

Lemma: If $H$ is connected, algebraic, and generated by 1-parameter algebraic subgroups, ~~and if~~ then any invariant measure is supported on $H$-fixed points ($H \curvearrowright V$ rationally).

Pf: It suffices to prove this for $H = \mathbb{R}$. By Poincaré Recurrence, $\mathrm{supp}(\mu)$ consists of nonwandering points.
↪

But by Rosenlicht, orbits are open in their closures.

$\Rightarrow$ no recurrence unless a fixed point.

Lemma: $G \curvearrowright G/H$ has a $G$-fixed point $\Leftrightarrow$ $H = G$

Pf: Assume $\exists$ a $G$-fixed point $g_0 H$, i.e., $\forall g \in G$: $g g_0 H = g_0 H$.

$\Rightarrow \forall g \in G, \exists h \in H$ such that

$$g g_0 = g_0 h \Rightarrow g_0^{-1} g g_0 = h.$$

$\Rightarrow g_0^{-1} G g_0 \subseteq H$

$\hookrightarrow$

Since $\dim(G) = \dim(g_0^{-1} G g_0)$,
$G = H$.

**Theorem** (Borel Density): Let $\Gamma \leq G$ be a lattice in a semisimple algebraic group $G$. Let $\overline{\Gamma}^Z$ denote the Zariski closure (smallest algebraic variety containing $\Gamma$). Then $\overline{\Gamma}^Z = G$.

Pf: Let $H = \overline{\Gamma}^Z$

For now assume $H$ is a subgroup. Then $\pi : G/\Gamma \to G/H$ is well defined $\Rightarrow$ $\pi_*(\mu)$ is a $G$-inv. prob. measure on $G/H$ $\Rightarrow$ $H = G$.

(Haar)

In general $\overline{\Gamma}^Z$ is not a subgroup.

The actions $G \curvearrowright G/Q$ are called <u>boundary</u> <u>actions</u>.

- They don't preserve a measure
- Non-recurrent.

<u>HW</u>: $Q = \begin{pmatrix} * & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & \vdots & & \vdots \\ 0 & * & \cdots & * \end{pmatrix}$ is a subgroup and ...

$\Rightarrow G/Q \cong \mathbb{R}P^{d-1}$

$G \curvearrowright G/Q$ is the projectivized action