KV- Topics
2/11/25

Howe-Moore / Decay of Matrix Coefficients / Mautner Phenomenon:

Let G be a simple Lie group and $\pi : G \to U(\mathcal{H})$ be a unitary representation ~~tote~~ without fixed points. If $g_n \longrightarrow \infty$ (escapes every compact set), then

$$\forall \phi, \psi \in \mathcal{H} : \langle \pi(g_n)\phi, \psi \rangle \to 0$$

Cor: If $G \curvearrowright (X, \mu)$ is (measure-preserving) ergodic and $H \leq G$ is a noncompact subgroup, then $H \curvearrowright (X, \mu)$ is ergodic.

Note: The corollary is false if μ is only quasi-invariant,

e.g. the boundary actions

e.g. $SL(2,\mathbb{R}) \curvearrowright S^1$

$\left(SL(d,\mathbb{R}) \curvearrowright S^{d-1} \right)$.

Pf of Cor (Assuming Howe-Moore)

Let $\mathcal{H} = L^2(X,\mu) / \text{constants} \cong L^2_0(X,\mu)$

$= \ker(\mathbb{E}_\mu)$,

then $\pi : G \to \mathcal{U}\left(L^2_0(X,\mu)\right)$

$$g \mapsto \left[f \mapsto f \circ g^{-1} \right].$$

$\pi|_H$ also has no H-fixed points,

for if ϕ were H-fixed,

$\hookrightarrow$

$$|\phi|^2 = \langle \phi, \phi \rangle = \langle \pi(h_n)\phi, \phi \rangle \to 0 \qquad \square$$

Lemma: Let G be simple and $A \subseteq G$ be $\mathbb{R}$-split Cartan. If $a \in A$ is nonzero, then for

$$U^+ = \langle \text{expanding root subgroups of } a \rangle$$

$$\left(\text{aka. } , \ \mathrm{lie}(U^+) = \bigoplus_{\lambda(a) > 0} E^\lambda \right)$$

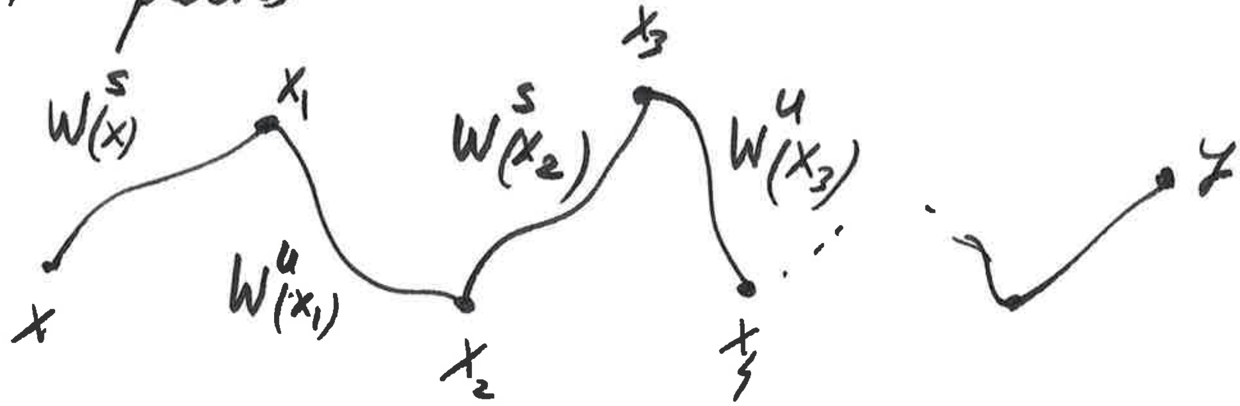
$$U^- = \langle \text{contracting root subgroups of } a \rangle,$$

$$\left(\text{aka } , \ \mathrm{lie}(U^-) = \bigoplus_{\lambda(a) < 0} E^\lambda \right)$$

$$\langle U^-, U^+ \rangle = G.$$

[Accessibility]

Remark: If $f: X \to X$ has stable/unstable manifolds, the accessibility class of $x \in X$ is the set of points y reached through finite s/u-paths



Another interpretation of the lemma: If $G \curvearrowright X$ and $g \in G$ has stable/unstable manifolds, accessibility classes are saturated by G-orbits.

$\star$

Pf of lemma: Let $H = \langle U^+, U^- \rangle$.

We'll show $H \lhd G$.

$\Updownarrow$

We'll show $\mathrm{Lie}(H)$ is an ideal in $\mathrm{Lie}(G)$.

Recall: $\mathrm{Lie}(G)$

(centralizer of Cartan)

$$= \mathrm{Lie}(A) \oplus \mathrm{Lie}(M) \oplus \bigoplus_{\lambda \in \Delta} \mathfrak{g}_\lambda$$

(Root space decomposition)

Key feature: If $X \in \mathrm{Lie}(A)$, $Y \in \mathfrak{g}_\lambda$

$\Rightarrow [X, Y] = \lambda(X) \cdot Y$

$\hookrightarrow$

Note: If $Z \in \mathrm{Lie}(M)$, then $[Z, \mathfrak{g}_\lambda] \subseteq \mathfrak{g}_\lambda$

$$\Rightarrow \mathrm{ad}_X \, \mathrm{ad}_Z (Y) = \mathrm{ad}_Z \, \mathrm{ad}_X (Y)$$

$$= \mathrm{ad}_Z (\lambda(X) \cdot Y) = \lambda(X) \cdot \mathrm{ad}_Z (Y)$$

$$\Rightarrow \mathrm{ad}_Z (Y) \in \mathfrak{g}_\lambda.$$

Since $\mathrm{Lie}(U^+)$, $\mathrm{Lie}(U^-)$ are sums of root spaces, it's immediate that $\mathrm{Lie}(A)$, $\mathrm{Lie}(M)$ have

$$[\mathrm{Lie}(A), \mathrm{Lie}(H)] \subseteq \mathrm{Lie}(H)$$

$$[\mathrm{Lie}(M), \mathrm{Lie}(H)] \subseteq \mathrm{Lie}(H)$$

Need to check $\forall \lambda \in \Delta$, $\lambda(a) = 0$

$$\Rightarrow [\mathfrak{g}_\lambda, \mathrm{Lie}(U^\pm)] \subseteq \mathrm{Lie}(U^\pm).$$

$\hookrightarrow$

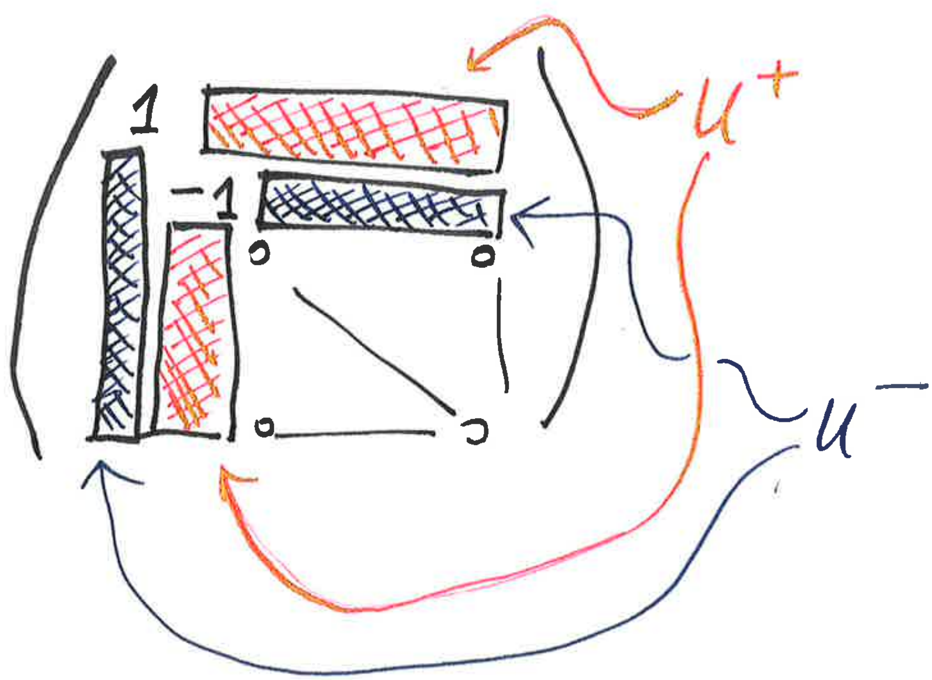
If $x \in \Delta$, then $[\mathfrak{g}_x, \mathfrak{g}_\lambda] \subseteq \mathfrak{g}_{x+\lambda}$.

Since $(x+\lambda)(a) = x(a)$, U^+, U^- is preserved by the vanishing weights (root).

Ex: $G = SL(10000, \mathbb{R})$

$X = \operatorname{diag}(1, -1, 0, \ldots, 0)$

$\rightsquigarrow$



(*) **Lemma**: If G semisimple, $g \in G$,

then $\exists$ $\mathbb{R}$-semisimple a

$\exists$ Ad-compact k

$\exists$ unipotent u

such that $g = aku$,

a, k, u pairwise commuting.

[Jordan decomposition (?) $\sim KAN$]

Pf of Decay of coeffs; special case:

$\boxed{\text{Case: Assume } g_n \in A}$ [To resolve in general case, use $G = KAK$]

Consider the roots $\Delta = \{\lambda_1, \dots, \lambda_e\}$

$\hookrightarrow$

Since g_n escapes every compact set, $\exists i$ such that ~~$\exists$~~ $\lambda_i(g_n) \to \infty$ as $n \to \infty$. (use a set of simple roots as a dual basis; e.g. in $SL(d,\mathbb{R})$, $t_{i+1} - t_i$).

Claim: If $\pi(g_n) \to E$, then $E\,\pi(u) = E,\ \forall u \in U_\lambda$.

Indeed, ~~[illegible]~~

$$\langle E\pi(u)\phi, \psi\rangle = \lim_{n\to\infty} \langle \pi(g_n)\pi(u)\phi, \psi\rangle$$

$$= \lim_{n\to\infty} \langle \pi(\underbrace{g_n u g_n^{-1}}_{\to e})\, \pi(g_n)\phi, \psi\rangle$$

$$= \lim_{n\to\infty} \langle \pi(g_n)\phi, \psi\rangle = \langle E\phi, \psi\rangle.$$

Similarly, if $u \in U_{-\lambda}$, $E^* \pi(u) = E^*$.

But $\ker E = \ker E^*$, since

$$|E\phi|^2 = |E^*\phi|^2.$$

Let $U^+ = \left\langle \begin{array}{c} \text{weights such that} \\ \lambda_i(g_n) \to \infty \end{array} \right\rangle$

$= U^+(a_0)$ by passing to a subseq.

Similarly $U^- = U^-(a_0)$.

But $\langle U^+, U^- \rangle = G$.

Now, the U^+-inv. functions and U^--inv. functions are perpendicular to $\ker(E)$.

(More next time.) $\boxed{\text{Want: } E = 0}$