KV - Topics
4/8/25

(Zimmer Ch. 9.1.2)

Theorem: Let G have property (T) and $G \curvearrowright (X,\mu)$ be a p.m.p action. If H is amenable and $\beta : G \times X \to H$ is a cocycle, then β is cohomologous to a cocycle taking values in a compact group.

"Compact cocycle"

Prop: Assume $G \overset{\alpha}{\curvearrowright} (X,\mu)$ is p.m.p and $\beta : G \times X \to H$ is a cocycle taking values in $H \subseteq GL(d,\mathbb{R})$. Then β is a compact cocycle iff $\exists$ function

$$\varphi : X \to L^2(H)_1 = \{ f \in L^2(H) \mid \|f\|_{L^2} = 1 \}$$

such that $\varphi(gx)(h) = \varphi(x)(\beta(g,x)^{-1}h)$

Pf: ($\Rightarrow$) Assume $\exists$ compact $K \subseteq H$ such that

$\tau : X \to H$ and $\bar{\beta} : G \times X \to K$ such that

$$\beta(g,x) = \tau(gx)^{-1} \bar{\beta}(g,x) \tau(x)$$

Fix $f_0 \in L^2(H)_1$ $(f_0 > 0)$, and let

$$f = \int_K f_0 \circ L_{k^{-1}} \, d\nu_K(k)$$

(Haar of K)

$\Rightarrow f$ is K-invariant,

wlog ~~####~~ $|f| = 1$.

Define $f_h = f \circ L_{h^{-1}}$, $\varphi(x) = f_{\tau(x)}$

$$\Rightarrow \varphi(x) \beta(g,x)^{-1} = f_{\tau(gx)} = \varphi(gx).$$

$\hookleftarrow$

($\Leftarrow$) Assume $\exists \varphi \in L^2(H)_1$ such that

$$\varphi(gx)(h) = \varphi(x)\left(\beta(g,x)^{-1} h\right)$$

Note: If $h \to \infty$ in H (i.e., escapes every compact subset), then $f \circ L_{h^{-1}} \to 0$ in $L^2(H)$ $\forall f \in L^2(H)$

In particular, $H \curvearrowright L^2(H)$ is tame in L^2-norm.

(orbits are open in their closures.)

Furthermore, $\forall f \in L^2(H), f \neq 0$, $Stab(f)$ is compact.

Now, following proofs from algebraic hull reductions,

G

$$\varphi \circ \alpha(g) = \varphi \circ L_{\beta(g,x)^{-1}}$$

(H-equivariance in bundle language)

fiber group: Unitary $(L^2(H))$

$\Rightarrow$ By tameness

$\mathrm{Im}(\varphi)$ lies in a single orbit

$\Rightarrow \exists$ Stab $(\varphi(x_0))$ - reduction.

<u>Lemma</u>: Let G be a topological group with a Haar measure μ. If $B \subseteq G$ is compact and $\mu(B) > 0$, then $\underbrace{B^{-1}B}$ contains a neighborhood of e.

$\{g^{-1}h \mid g,h \in B\}$.

Pf: Observe: $x \in B^{-1}B \Leftrightarrow (Bx) \cap B \neq \phi$.

$\Rightarrow$ Want: $\{x \in G \mid (Bx) \cap B \neq \phi\}$ is open at e.

Choose an open set W such that $B \subseteq W$ and $\mu(W) < 2\mu(B)$.

Since B is compact, $\exists$ symmetric neighborhood U of e such that $xU \subseteq W$, $\forall x \in B$.

$\Leftrightarrow Bx \subseteq W \quad \forall x \in U$

$(\Leftrightarrow BU \subseteq W)$

Then $\mu(Bx) = \mu(B)$, $\leftarrow$ (b/c μ is right invariant)

G.

so $(Bx) \cap B = \emptyset$

$\Leftrightarrow \mu((Bx) \cap B) = 2\mu(B) > \mu(W)$

But $(Bx) \cup B \subseteq W$

$\Rightarrow \mu((Bx) \cup B) < \mu(W)$.,

a contradiction.

Pf of Theorem: By Prop., it suffices to find a β-equivariant function

$\varphi : X \to L^2(H)$ (that is nonzero)

$\varphi(gx)(h) = \varphi(x)(\beta(g,x)^{-1}h)$

Define a rep of G by

$G \curvearrowright$

$$\Big[\big[\sigma(g)\varphi\big](x)\Big](h)$$

$$= \varphi(g^{-1}x)\big(\beta(g,x)h\big).$$

- φ is σ-invariant iff φ is β-equivariant. (Exr)

We'll find (ε, K)-invariant vectors $\forall\, \varepsilon, K$.

$\boxed{\text{Idea}}$ Look at $\varphi(x) \equiv f \in L^2(H)$

Indeed, we'll choose f such that

$$|f - f \circ L_{h^{-1}}| < \varepsilon \text{ for } \underbrace{h \in \mathrm{Im}\big(\bar{\beta}|_{K\times X}\big)}_{\text{sticky part}}.$$

↳

$$\Rightarrow |\sigma(g)\varphi - \varphi|^2 = \int_X \left|[\sigma(g)\varphi](x) - \varphi(x)\right|^2_{L^2(H)} d\mu(x).$$

$$= \int_X \int_H \left(f\left(\beta(g,x)^{-1} h\right) - f(h) \right)^2 d\nu_H(h)\, d\mu(x)$$

(Haar on H)

[redacted] $< \int_X \varepsilon^2 \, d\mu(x) = \varepsilon^2$

• How to fix the sticky part?

Let $S(g,A) = \{ x \in X \mid \beta(g,x), \beta(g,x)^{-1} \in A \}$

for $g \in G$, [redacted] $A \subseteq H$.

$\exists A \subseteq H$ such that

[redacted]

$\hookrightarrow$

$$(\nu_G \times \mu)\left(\underbrace{\left\{(g,x)\in K\times X \;\middle|\; \begin{array}{l}\beta(g,x)\in A,\\ \beta(g,x)^{-1}\in A\end{array}\right\}}_{\text{"= set of all } (g,\, S(g,A))\text{"}}\right) \geq \left(1-\frac{\varepsilon}{3}\right)\nu_G(K)$$

Fubini $\Rightarrow$ $K_0 := \left\{ g\in K \;\middle|\; \mu(S(g,A)) > 1-\frac{\varepsilon}{3} \right\}$

has positive ν_G-measure.

K_0 is symmetric in G

$\Rightarrow$ $K_0^2 = K_0^{-1}K_0$ has an open neighborhood of e

$\forall g \in W \subseteq K_0^2$, $\mu(S(g,A^2)) > 1-\frac{2\varepsilon}{3}$.

$\hookrightarrow$

Cover K with finitely many translates of W. Pick B such that $\mu\left(S(g_i, B)\right) > 1 - \frac{\varepsilon}{3}$ for every center g_i of translates.

Then $\mu\left(S(g, A^2B)\right) > 1-\varepsilon$ $\forall g \in K$ (by the cocycle identity)

This is enough to make the proof work:

Choose $f \in L^2(H)$ such that $|f - f \circ L_{h^{-1}}| < \varepsilon$ $\forall h \in A^2B$, f bounded.