KV-Topics
4/3/25

Zimmer - Ch. 7 - Property (T):

Def: Fix a locally compact top. group G, and a unitary rep. $\rho : G \to \mathcal{U}(\mathcal{H})$. If $\varepsilon > 0$ and $K \subseteq G$ compact subset, a unit vector $v \in \mathcal{H}$ is called (ε, K)-invariant iff

$$|\rho(g)v - v| < \varepsilon, \forall g \in K.$$

"invariant vector" $\longleftrightarrow$ "$(0, G)$-invariant.

Ex: $G = \mathbb{Z}^d$

$\mathcal{H} = L^2(\mathbb{Z}^d, \mu)$,

μ = counting measure.

$\rho(n) f(m) = f(m-n)$

$\hookrightarrow$

Aside: Our applications will be $\mathcal{H} = L^2(X, \mu)$

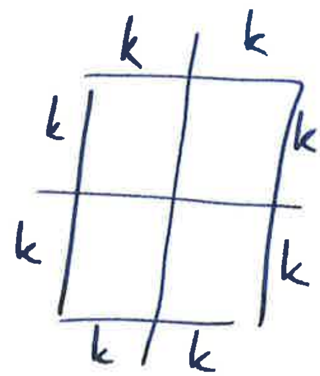
$G \curvearrowright X$

$\rho(g) f = f \circ g^{-1}$

Koopman.

There are no ρ-invariant vectors, as invariant $\Rightarrow$ constant $\Rightarrow 0$.

Let $f_k = \frac{1}{\sqrt{(1+2k)^d}}\, \chi_{[-k,k]^d}$



Case: $K = \{\pm e_i \mid i = 1, \dots, d\}$

$$\Rightarrow |\rho(e_i) f_k - f_k|^2 = \frac{2}{2k+1} \xrightarrow{k \to \infty} 0$$

Ex: $G = \mathbb{Z}$, $\mathcal{H} = L^2_0(\mathbb{R}/\mathbb{Z}, \text{Leb})$

$\uparrow$ $L^2/\mathbb{R}$

$\rho(n) f = f \circ R_\theta^{-n}$,

$\theta \notin \mathbb{Q}$.

$\exists$ $(\varepsilon, \{q\})$-invariant vectors

q : partial quotients.

(more of an analogy)

Def: If G is a countable group, a Følner sequence is a collection of finite subsets $F_1 \subseteq F_2 \subseteq \cdots \subseteq F_n \subseteq \cdots$ such that

- $\bigcup_n F_n = G$
- $\forall g \in G : \lim_{n \to \infty} \dfrac{\#(gF_n \,\Delta\, F_n)}{\#(F_n)} = 0$

(the ratio of ∂/interior goes to 0.)

Remark: When not countable, use Haar instead of #, but be careful (nets instead of sequences etc.)

Exr: Show that if G has a Følner sequence, then $\forall K, \varepsilon$, $\exists (\varepsilon, K)$-invariant vectors

Def: G is called <u>amenable</u> if it has a Følner sequence.

Def: G has <u>(Kazhdan's) Property (T)</u> if any unitary rep. with (ε, K)-invariant vectors $\forall\, \varepsilon, K$, has an invariant vector.

<u>Property (T)</u>: If ρ has no invariant vectors, then $\exists (\varepsilon, K)$: $\nexists$ (ε, K)-invariant vectors

($\Rightarrow$ $\nexists$ simultaneously recurrent functions).

Ex: Abelian, nilpotent, solvable countable groups with infinitely elements are amenable but don't have property (T).

Theorem (Kazhdan): If $\operatorname{rank}_{\mathbb{R}}(G) \geq 2$, G semisimple, then G has property (T). The same is true for its lattices.

Theorem: G is amenable with property (T) $\iff$ G is compact.

Pf: ($\Rightarrow$) Amenable $\Rightarrow \exists$ almost-invariant vectors in $L^2(G,\mu)$ (by Følner sets)

Prop. (T) $\Rightarrow \exists$ inv. vector in $L^2(G,\mu)$

$\Rightarrow \mu(G) < \infty \Rightarrow G$ compact.

↳

($\Leftarrow$) <u>Amenability</u>: We construct F_n explicitly by $F_n = G$, $\forall n$.

<u>Property (T)</u>: Let G be compact and ρ have almost invariant vectors.

<u>Note</u>: $\forall v \in \mathcal{H}$, $\bar{v} = \int_G \rho(g) v \, d\mu(g)$ is G-invariant.

<u>Danger</u>: $\bar{v} = 0$, $\forall v$.

Use almost ~~xxxx~~ invariance to pick $v \in \mathcal{H}$ such that

$$|\rho(g)v - v| < \frac{1}{2} \quad \forall g \in G.$$

$$\Rightarrow |\bar{v} - v| = \left| \int_G \rho(g) v \, d\mu(g) - v \right|$$

$$\leq \frac{1}{2}. \quad \Rightarrow \quad \bar{v} \neq 0.$$

(Zimmer, Ch. 8)

Theorem: Let $G \curvearrowright (X, \mu)$ is p.m.p. and $\beta : G \times X \to H$ be a cocycle taking values in an amenable group. If G has property (T), then β is cohomologous to a cocycle taking values in a compact group.

(Pf next time)

(Other) Consequences of Property (T):

- Γ countable w/ property (T) $\Rightarrow$ Γ is finitely generated.
- Γ countable w/ property (T) $\Rightarrow \left| \Gamma / [\Gamma, \Gamma] \right| < \infty$.