KV-Topics
4/22/25

Actions in Critical Dimension

(Brown-RH-Wang)

$\Gamma \curvearrowright M$ C^∞-action, faithful, not PMP.

$\Gamma \leq G$ lattice,

dim(M) = critical = maximal codim of a parabolic in G.

Step 0: Suspension.

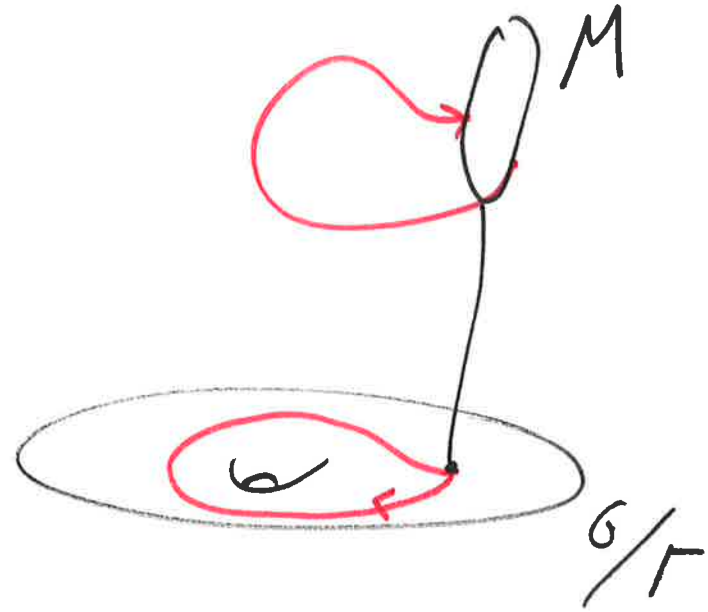
$\Gamma \curvearrowright M$.

$\tilde{M} = G \times M \curvearrowleft \Gamma$

$(g, x) \cdot \gamma = (g\gamma, \gamma^{-1}x)$

$\hat{M} = G \times M / \Gamma$

$$\hat{M} \xrightarrow{\;p\;} G/\Gamma$$



$G \curvearrowright \hat{M}$ does not preserve a prob. measure.

On the other hand $P \leq G$, the minimal parabolic always preserves a measure, since it is solvable, hence amenable.

Ex: $G = SL(d, \mathbb{R})$

$$P = \begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ 0 & & * \end{pmatrix} \qquad G \curvearrowright S^{d-1}$$

$\hookrightarrow$

The P-inv. measures are: δ_{e_1}.

Exponents of the diagonal action rel δ_{e_1}?

S^{d-1} is a double cover of $\mathbb{P}^{d-1}$, hence can use the projective coordinates to find evals of $da(e_1)$ for $a \in A = \mathbb{R}$-split Cartan.

$$[x:y:z]\Big|_{x=1} \longmapsto \left[e^{t_1}x : e^{t_2}y : e^{t_3}z\right]\Big|_{x=1}$$

$$= \left[1 : e^{t_2-t_1}y : e^{t_3-t_1}z\right]$$

$\Rightarrow$ $\boxed{\text{Exponents} = t_2 - t_1,\ t_3 - t_1.}$

Notice: These are exactly the roots complementary to:

$$Q = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} = \mathrm{Stab}(e_1)$$

$$\left(S^{d-1} = G/Q \right).$$

Let μ be a P-invariant measure on $\hat{\mathcal{M}}$ such that $P * \mu = \text{Haar}$.

Apply Oseledets to the A-action to get two families of exponents:

$\hookrightarrow$

$$T\hat{M}$$
$$\parallel$$
(base) TG = root space decomposition
$$\oplus$$
(fiber) $TM = \bigoplus_{\lambda \in \Delta_F} E_f^\lambda$

These may interact.

<u>Pesin theory (for abelian actions [B-RH-W])</u>:

$\exists$ a.e. defined (local) foliations W^λ for $\lambda \in \Delta$ such that $T_x W^\lambda(x) = E^\lambda(x)$ for μ-a.e. x.

Furthermore, it is possible to disintegrate μ into measures $\{\mu_x^\lambda\}_x$ supported on precompact open subsets of $W^\lambda(x)$.

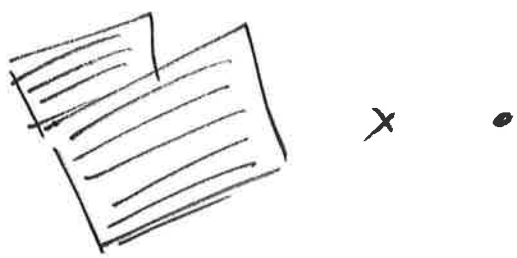
Ex: $\mathbb{Z}^2 \curvearrowright \mathbb{T}^2 \times \mathbb{T}^2$, $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.

$$(m,n)\cdot(x,y) = (A^m x, A^n y)$$

$$\Delta = \{\pm \chi dx, \pm \chi dy\},$$

χ = log eval of A.

Pick a Markov partition of A, then partition into u-leaves.



• In general, a measurable partition ξ is subordinate to W if:

$\hookrightarrow$

(i) $\xi(x) \subseteq W(x)$ a.e. x

(ii) $\xi(x) \subseteq W(x)$ contains an open set containing x, a.e. x.

(iii) $\overline{\xi(x)}^{W(x)}$ is compact in $W(x)$.

$$h_\mu(a|W^\lambda) = \sup\{h_\mu(a, \xi \vee \eta) \mid \eta \text{ any mble partition}\}.$$

Intuition:

If $h_\mu(a|W^\lambda) = b$, then

$$\lim_{n\to\infty} -\frac{1}{n} \log \frac{d(a^n_* \mu^\lambda_x)}{d\mu^\lambda_{a^n x}} = b.$$

e.g. if $\mu^\lambda_x = \delta_x$, then $h_\mu(a|W^\lambda) = 0$.

if $\mu_x^\lambda = \mathrm{Leb}$, then $h_\mu(a \mid W^\lambda)$

$= \sum$ Lyap exp.

Thm (Ledrappier–Young, B–RH–W):

$$h_\mu(a) = \sum_{\lambda(a)>0} h_\mu(a \mid W^\lambda)$$

Thm: In the suspension setting,

if λ is a root θ, then

(Abramov type formula)

$$h_\mu(a \mid W^\lambda) = h_{Haar}(a \mid W_b^\lambda) + h_\mu(a \mid W_f^\lambda)$$

Theorem: If λ is a root of G, μ is invariant $U_\lambda \Leftrightarrow \mu_x^\lambda$ disintegrates as Leb along U_λ-orbits in W^λ.

This theorem implies that if λ is not a fiber exponent, μ is invariant under U_λ.

We assumed μ is P-inv. but not G-inv.

If # of resonant weights $< r$ = crit. dim, then μ is invariant under a parabolic of codim $< r \Rightarrow$ all of G.

$\Rightarrow$ The fiber exponents are exactly these "missing" from a maximal parabolic.

$Q = \mathrm{Stab}(\mu)$ = maximal parabolic.

Last trick:

$$\hat{M} = \overbrace{G \times M}^{\tilde{M}} \big/ \Gamma^{\Delta}$$

lift μ to $\tilde{\mu}$ on $\tilde{M}$.

$$G \curvearrowright \tilde{M} \curvearrowleft \Gamma$$

$$\big\downarrow$$

$$Q \curvearrowright \tilde{M} \curvearrowleft \Gamma$$

$$Q \backslash (G \times M) \xrightarrow{\pi} M$$

$$\big\downarrow$$

$$Q \backslash G$$

π is a measurable iso./conjugacy between Γ actions.