KV-Topics
4/17/25

Last Time: $G \curvearrowright X$ totally partially hyperbolic, volume-preserving, accessible $\Rightarrow$ On each E^λ, $\exists\, |\cdot|$ such that

$$|da(v)| = e^{\lambda(a)}|v|, \quad \forall v \in E^\lambda, \ a \in A \subseteq G$$

$\mathbb{R}$-split Cartan

(Δ = coarse Lyap exps)

(☺)

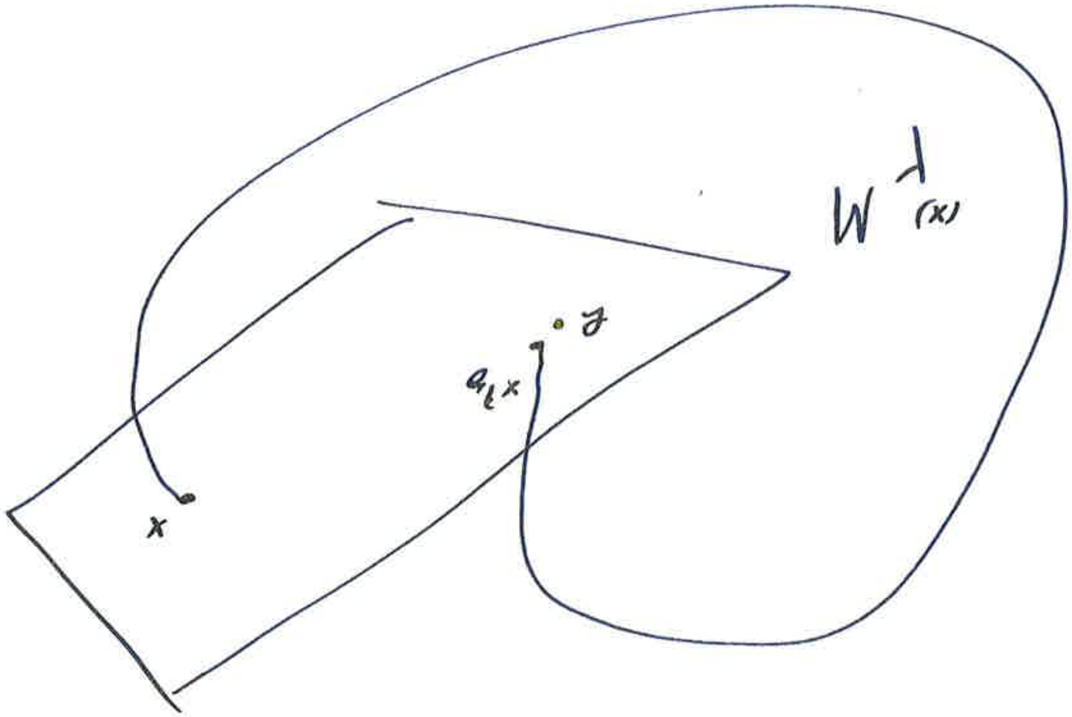
Pre-Lemma: $\forall \lambda \in \Delta$, $\ker(\lambda)$ has a dense orbit

Pf: $\exists\, a \in \ker(\lambda)$ such that a is $(E^c \oplus E^\lambda \oplus E^{-\lambda})$-PH and accessible.

Lemma: $\mathrm{Isom}(W^\lambda(x))$ acts transitively on $W^\lambda(x)$.

(relative to the above $|\cdot|$)

Pf: Because the metrics are C^0, it suffices to prove this $\forall x$ w/ dense $\ker(\lambda)$-orbit.



$a_k x \to y$
$a_k \in \ker(\lambda)$

$a_k : W^\lambda(x) \to W^\lambda(y) = W^\lambda(x)$
converges to an isometry

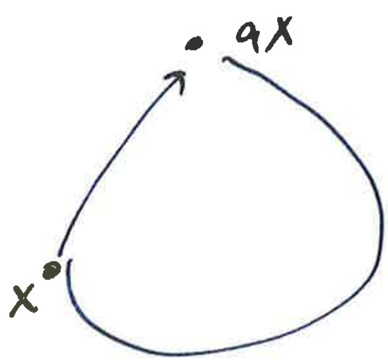
"higher-rank trick"

Lemma: $\mathrm{Isom}(W^\lambda(x))$ has a unique nilpotent subgroup acting simply transitively on $W^\lambda(x)$.

Warning: In the case of $\mathbb{H}^2$, there is no simply transitive subgroup $N \leq \mathrm{Isom}(\mathbb{H}^2)$ that is nilpotent.
(or $\mathbb{H}^n$)
(or normal)

(Exercise)

Pf: $a \notin \ker(\lambda)$



$b_k a x \to x$,

$b_k \in \ker(\lambda)$

a fixed, b_k isometric

$\Rightarrow b_k a : W^\lambda(x) \to W^\lambda(x)$ converges.

G

Let $\varphi = \lim_{k \to \infty} b_k a$.

$$\Phi : \mathrm{Isom}(W^{\lambda}(x)) \circlearrowleft$$

$$\Phi(f) = \varphi \circ f \circ \varphi^{-1}$$

is an automorphism.

Φ induces a splitting

$$\mathrm{Lie}(\mathrm{Isom}(W^{\lambda}(x))) = V \overset{(*)}{\oplus} \{\text{unitary eigenspaces}\}$$

V contracting under Φ.

$$\{\text{unitary eigenspaces}\} = \mathrm{Stab}(x)$$

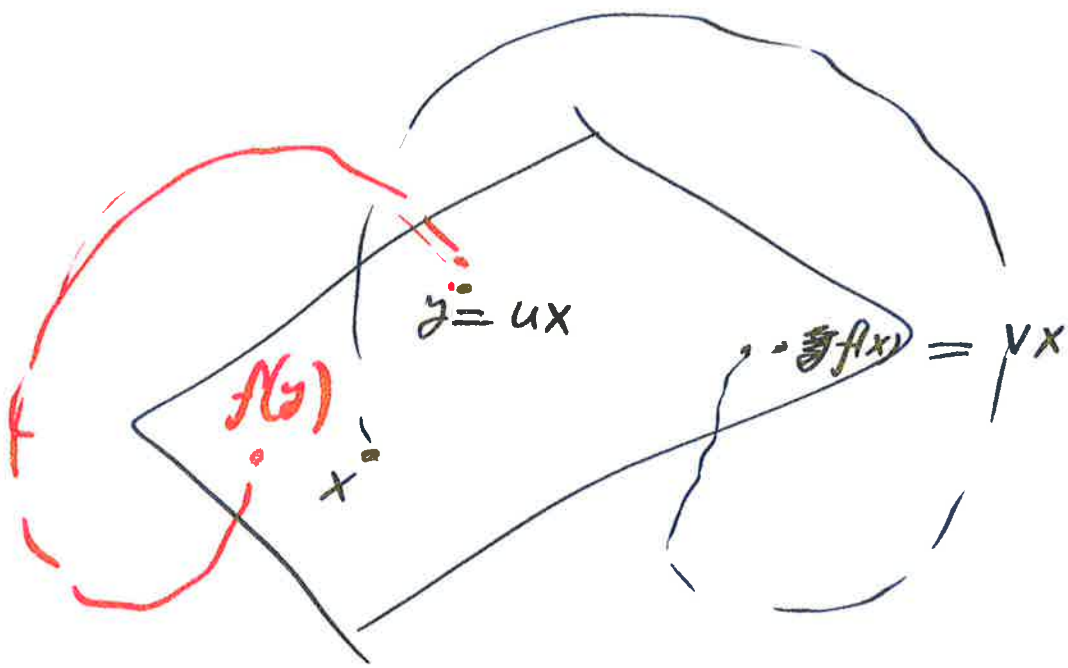
$$V = \text{translation subalgebra.}$$

Exr: a) subalgebras w/ contracting autos are nilpotent

b) subalgebras w/ nonexpanding autos are solvable.

Let $^{\lambda}N_x$ = simply transitive nil subgroup of $\mathrm{Isom}(W^{\lambda}(x))$.

<u>Warning</u>: This is <u>not</u> the action we're looking for.



$u \in N^{\lambda}$

$a_k x \to f(x)$

$$f(y) = \lim_{k\to\infty} a_k y = \lim_{k\to\infty} a_k u x$$

$$= \lim_{k\to\infty} u a_k x = uvx$$

$\Rightarrow \quad ux \longmapsto uvx$

This acts like right translation on $W^\lambda x$.

(i.e. ${}^\lambda N_x$)

In Lie groups, we can build the left translations from the right by considering RIVF's.

Idea: Consider vector fields invariant under ${}^\lambda N_x$ on $W^\lambda(x)$, $\mathfrak{X}(W^\lambda(x))$

- Build an abstract Lie algebra n^λ which is isomorphic to $\mathfrak{X}(W^\lambda(x)) \quad \forall x$.

- Build a compact fiber bundle whose fibers are $\{\varphi : n^\lambda \to \mathfrak{X}(W^\lambda(x)),\ \varphi \text{ is an isometric alito.}\}$

Annoying and technical work shows that the bundle is a principal K-bundle, where $K = \{\text{isometric autos of } \pi^{\lambda}\}$, and the actions of A, K, N^{λ} all lift and/or are well-defined, satisfying

1) $[A, K] = e$
2) $[A, N^{\lambda}] \subseteq N^{\lambda}$ $\Rightarrow$ $au = [\psi_{\lambda}(a)u]\, a$, $\quad \psi_{\lambda} : A \to \mathrm{Aut}(N^{\lambda})$
3) $[K, N^{\lambda}] \subseteq N^{\lambda}$

$\Downarrow$

eigenvalues of ψ_{λ} are proportional to $e^{\lambda(a)}$

(In the algebraic model, H will be the group gen. by A, K, N^{λ})

Moving Forward:

$$K, A, N^{\perp} \curvearrowright \hat{X}$$ may not be accessible!

structure group

$$\downarrow$$

$$A \triangle X$$ accessible

Ex: $\Gamma = \mathbb{Z}$-points in some $\mathbb{Q}$-reps of $SL(3,\mathbb{R}) \times SU(3)$ (restriction of scalars)

$$SU(3) \Big\backslash \Big(SL(3,\mathbb{R}) \times SU(3)\Big) \ltimes_{\rho} \mathbb{R}^N \Big/ \Gamma^{\Delta} \ltimes_{\rho} \mathbb{Z}^N = X$$

carries an $SL(3,\mathbb{R})$ action.

$$\hat{X} = \Big(SL(3,\mathbb{R}) \times SU(3)\Big) \ltimes_{\rho} \mathbb{R}^N \Big/ \Gamma^{\Delta} \ltimes_{\rho} \mathbb{Z}^N$$

Path Groups:

Idea: If $U_1, \dots, U_\ell \leq G$ generate G, then $\exists$ onto homomorphism

$$\ker\left(U_1 * U_2 * \cdots * U_\ell \twoheadrightarrow G\right)$$

(free product)

$=$ relations in the generators in U_i

$=$ Stab(e) for

$$U_1 * U_2 * \cdots * U_\ell \curvearrowright G$$

by translations

Thm: $\langle K, A, N^{\lambda_1}, \ldots, N^{\lambda_\ell} \rangle$ is a Lie group

iff $\mathrm{Stab}_{(K \times A) \ltimes (N^{\lambda_1} * \cdots * N^{\lambda_\ell})}(x)^{\circ}$

is independent of x.

iff the contractible cycles in the groups K, A, N^{λ} are independent of x.

Step 1. All cycles can be written as concatenations of stable cycles and symplectic cycles.

(Easy part)

Step 2: Symplectic cycles:

relations in $N^{\lambda} * N^{-\lambda}$

$a \in \ker(\lambda)$

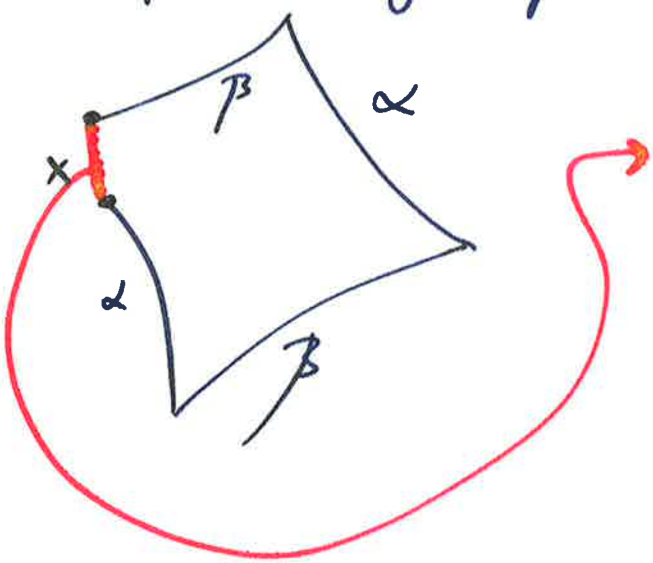
same cycle.

Step 3: Stable cycles: (Hard part)

(aka geometric commutators)

inspired by $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$.

Lift to group level



This is generated by elements in $N^{t\alpha+s\beta}$, $t,s\in\mathbb{R}_{>0}$

$$\gamma = t\alpha + s\beta$$

$$\rho_\gamma^{\alpha,\beta}(u,v,x) = \gamma\text{- component of } [u,v].$$

Thm: $\rho_\gamma^{\alpha,\beta}$ is a polynomial independent of x.

$\square$.