KV-Topics
4/15/25

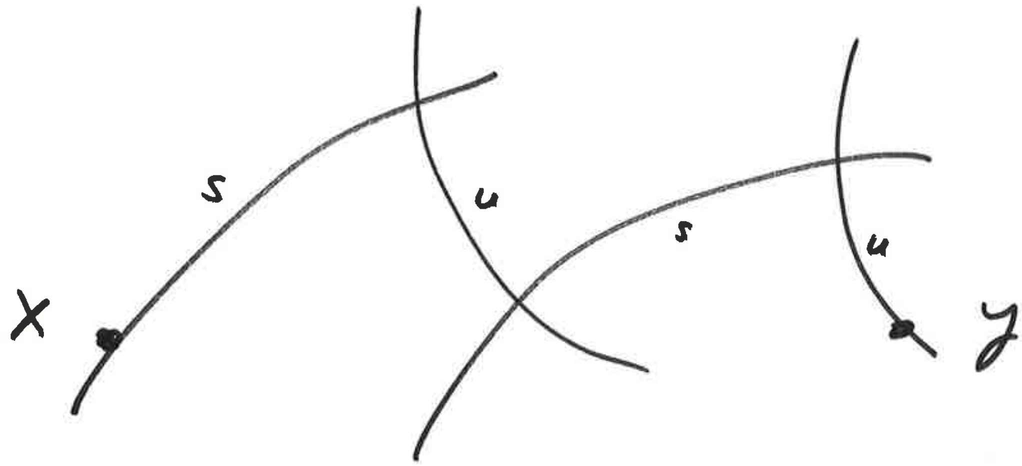
DSVX :

Def : $\mathbb{R}^k \curvearrowright X$ C^∞ action.

- $a \in \mathbb{R}^k$ is E^c-partially hyperbolic if $\exists$ splitting $TX = E_a^s \oplus E^c \oplus E_a^u$
- Let $\mathcal{PH}$ denote the set of E^c-partially hyperbolic elements in $\mathbb{R}^k$.
- $\mathbb{R}^k \curvearrowright X$ is totally partially hyperbolic if $\overline{\mathcal{PH}} = \mathbb{R}^k$
- $G \curvearrowright X$ is TPH if the restriction to its $\mathbb{R}$-split Cartan is TPH.

If one $\mathbb{R}$-split Cartan is TPH, then all $\mathbb{R}$-split Cartans are TPH.

Def: $f: X \to X$ is called <u>accessible</u> if $\forall x, y \in X$, $\exists$ su-path connecting x, y.



- $G \curvearrowright X$ is <u>accessible</u> if $\exists a \in A \subseteq G$ (split Cartan) such that a is E^c-PH and is accessible.

Theorem (DSVX): If $G \curvearrowright X$ is TPH, accessible and volume preserving, then $G \curvearrowright X$ is finitely covered by an action $G \curvearrowright {}_K\backslash^H/_\Gamma$ by translations, for G simple, $\mathrm{rank}_{\mathbb{R}}(G) \geq 2$.

Strategy: Use rble Zimmer, then upgrade using uniform structures.

Coarse Lyapunov Foliations:

Lem: If $a_1, \dots, a_\ell$ are E^c-PH elements, then $\hat{E} = E^s_{a_1} \cap E^s_{a_2} \cap \dots \cap E^s_{a_\ell}$ is a Hölder integrable distribution.

- $\hat{E} = \bigcap_{i=1}^{\ell} E^s_{a_i}$ is a (uniform) coarse Lyapunov distribution if $\forall a \in \mathcal{PH}$,

$$E^s_a \cap \hat{E} = \hat{E} \text{ or } \{0\}.$$

Exercise: Show that for homogeneous actions, $\forall\ \hat{E}\ \exists!\ \chi : \mathbb{R}^k \to \mathbb{R}$ such that $\hat{E} = \bigoplus_{c>0} E^{c\chi}$, where $E^{c\chi} = c\chi$ Ad-eigenspace.

Def: Fix a coarse Lyapunov foliation W with distribution E. Let $\mathcal{E}(E) \subseteq \mathbb{R}^k$ be the set of $a \in \mathbb{R}^k$ such that $da|_E$ is uniformly expanding. Similarly $\mathcal{C}(E)$ is contracting elements

Facts: • $\mathcal{E}(E)$ and $\mathcal{C}(E)$ are both convex (they are cones)

• $\overline{\mathcal{E}(E) \cup \mathcal{C}(E)} = \mathbb{R}^k$ by TPH condition.

Hahn-Banach Theorem: $\exists$ hyperplane H_E separating $\mathcal{E}(E)$ and $\mathcal{C}(E)$.

By TPH, $\mathbb{R}^k = \mathcal{E}(E) \sqcup H_E \sqcup \mathcal{C}(E)$.

For each E, choose a $\boxed{\chi : \mathbb{R}^k \to \mathbb{R}}$ such that $\mathcal{E}(E) = \{a \mid \chi(a) > 0\}$ and

$\mathcal{C}(E) = \{a \mid \chi(a) < 0\}$,

$\ker(\chi) = H_E$

<u>Lemma</u>: $\forall a \in \mathbb{R}^k$:

$da|_{E^c}$ has subexponential growth, if there are at least two independent coarse Lyapunov exponents.

Coarse Lyap. exponent $\in (\mathbb{R}^k)^* / \mathbb{R}_+$.

<u>Pf</u>: Fix a coarse exponent χ and choose $a \in \ker(\chi)$.

<u>Claim</u>: $da|_{E^c}$ has subexponential growth.

$\exists a'$ close to a: a' is E^c-PH.

$$\left| da'|_{E^c} \right| < \left| da|_{E^c} \right| < 1+\varepsilon$$

$$\Big\downarrow \; a' \to a$$

$$\left| da'|_{E^c} \right| \approx 1 .$$

<u>Invariance Principle</u> (Ledrappier, Wilkinson, Avila-Santamaria-Viana, Kalinin-Sadovskaya):

(mble ⇒ cts Livsic)

<u>Morally</u>: Measurable solution with only 1 exponent

+

accessibility

$\Downarrow$

continuous solution.

For a matrix cocycle.

Zimmer $\Rightarrow$ $\exists$ mble ($\rightsquigarrow$ continuous.) metrics on E^{χ} such that $|da(v)| = e^{\chi(a)} \cdot |v|$, $\forall v \in E^{\chi}$.

Here, χ are the weights of the rep of G that appears in Zimmer.

Pick $a \in \ker(\chi)$, so $da|_{E^{\chi}}$ is a matrix cocycle with only zero exponents.

<u>Claim</u>: $a \in \ker(\chi)$ can be chosen such that a is $(E^{c} \oplus E^{\chi} \oplus E^{-\chi})$-PH.
Furthermore, such an a is accessible.

"super-accessibility"

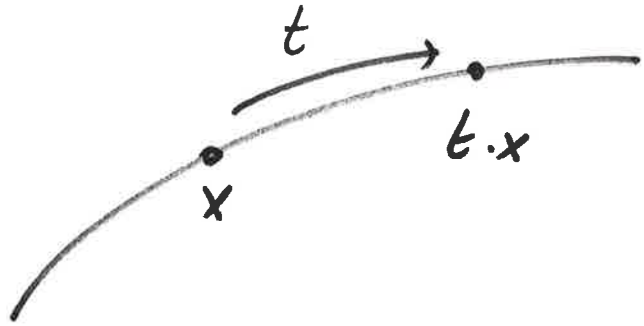
Pf: For first claim, just choose $a \in \ker(x)$ such that $a \in \ker(\beta)$, $\beta \not\propto x$.

For accessibility, note that the G orbits are contained in a-accessibility classes. (If x is a root, use that $\langle U_+, U_- \rangle = G$ as in Howe-Moore).

Lemma (Exr): If x is a weight of a rep $\rho: G \to GL(N, \mathbb{R})$, then $\exists$ root β such that $w_\beta(x) \not\propto x$.

(element of Weyl group)

- When coarses are all 1-dim, and oriented, define an $\mathbb{R}$-action with orbits W^x: (S-V)



- What happens $\dim(E^x) > 2$?
  Even if each leaf is isometric to $\mathbb{R}^2$, $(t_1, t_2)\cdot x$?

Ex: $\mathbb{R}^{d-1} \curvearrowright K \backslash SL(d,\mathbb{C}) / \Gamma$,

$$K = \left\{ \left(e^{i\theta_1}, \ldots, e^{i\theta_d} \right) \;\middle|\; \sum_{i=1}^{d} \theta_i = 0 \right\}$$

$\Rightarrow$ coarse Lyapunovs: $\begin{array}{cc} & j \\ i & \begin{pmatrix} & \square \\ & \end{pmatrix} \end{array}$,

parameterized by

$$U_{ij} = \left\{ \begin{array}{cc} & j \\ i & \begin{pmatrix} 1 & & z \\ & \ddots & \\ & & 1 \end{pmatrix} \end{array} \;\middle|\; z \in \mathbb{C} \right\}$$

but U_{ij} does not normalize K.

$\Rightarrow$ U_{ij} does not act on $K \backslash SL(d,\mathbb{C}) / \Gamma$.

$\hookrightarrow$

Case $d=2$:

$\mathbb{R} \curvearrowright S^1 \backslash SL(2,\mathbb{C}) / \Gamma$

$SL(2,\mathbb{C})$ = frame bundle to $\mathbb{H}^3$.

Horospherical action is <u>not</u> defined on $T\mathbb{H}^3/\Gamma$.

(most delicate part of the argument)

- We'll define a principal bundle with compact structure group to which the action lifts.
- On this bundle, there will be nil-group actions parameterizing each W^x.